Universes in simplicial homotopy type theory

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CHAPMAN UNIVERSITY

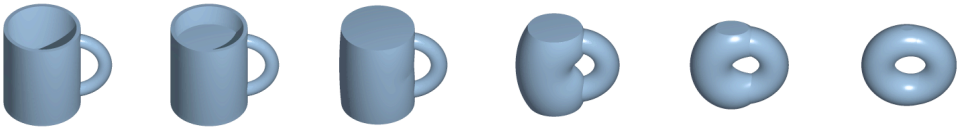



Topos Institute Colloquium, Oct 29, 2026

[1]Kindly supported by the Fletcher Jones Foundation.

A shift in foundations

Mathematics in the 21st century is fundamentally concerned with *deformations* $\rightsquigarrow$ Homotopy theory is primary



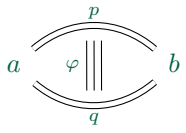
Animated image created by: Lucas Vieira (LucasVB), `https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif`

*Instead of **sets**, clouds of discrete elements, we envisage some sorts of vague **spaces**, which can be very severely deformed, mapped one to another, and all the while the specific space is not important, but only the space **up to deformation**. I am pretty strongly convinced that there is an **ongoing reversal** in the collective consciousness of mathematicians: the [. . .] **homotopical picture** of the world becomes the **basic intuition***

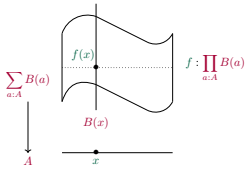
—Y. Manin ’09 (emphases mine)

HoTT and the Curry–Howard–Voevodsky interpretation[2]

Homotopy type theory	Homotopy theory
A	space/homotopy type
$x : A$	point
$\mathbf{0}$,$\mathbf{1}$	$\emptyset$, $*$
$A + B$	coproduct space
$A \times B$	product space
$A \to B$	function space
$x : A \vdash B(x)$	fibration
$x : A \vdash b : B(x)$	section
$\Sigma_{x:A} B(x)$	total space
$\Pi_{x:A} B(x)$	space of sections
$p : (x =_A y)$	path $x \rightsquigarrow y$ in A



Identity types



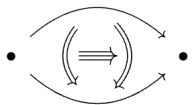
Dependent types as fibrations

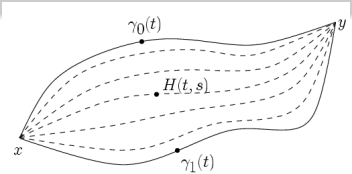
[2]Table based on: Emily Riehl *The synthetic theory of ∞-categories vs the synthetic theory of ∞-categories*, Presentation at Vladimir Voevodsky Memorial Conference, IAS, Princeton, NJ, USA, 2018.

∞-category theory

∞-categories:

- **higher morphisms** and **weak composition**



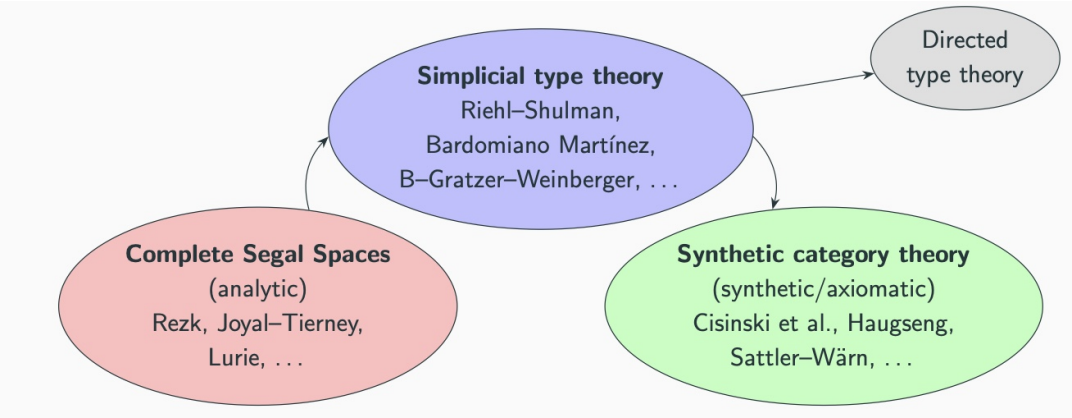


Created with xfig by Yonatan,
`https://commons.wikimedia.org/wiki/File:Homotopy_curves.png`

- Applications: Derived algebraic geometry, stable homotopy theory, topological quantum field theories, higher rewriting, ...
- More to come at Topos Colloquium by Ayala, Ozornova, Rovelli ...
- Dream: $(\infty, 1)$-category theory should be like 1-category theory, but up-to-homotopy.
- Problems in **analytic**, set-theoretic foundations: heavy encoding and not homotopy-invariant. Can we do better?
- **Yes, in synthetic foundations—using simplicial homotopy type theory!**

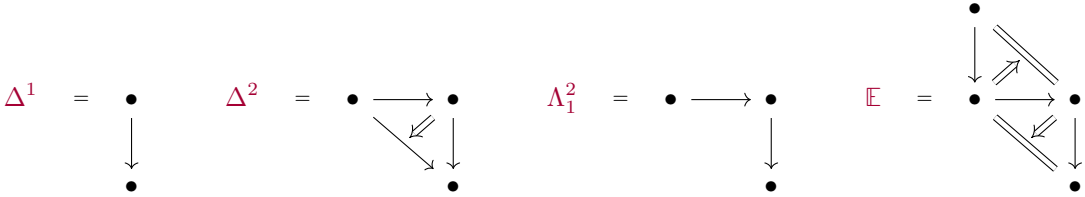
Functoriality and naturality for free! Reduction to finite-dimensional arguments
Proofs less model-dependent Verification via computer
∞-category theory for (under)graduates [Rie23] @ ICERM '26

Simplicial homotopy type theory as a bridge



Simplicial homotopy type theory

- **Question:** How to extend HoTT to capture ∞-categories?
- **Answer:** (after Riehl–Shulman)**:** HoTT + interval $\mathbb{I} := \Delta^1$







Higher Structures 1(1):116–193, 2017.

HIGHER STRUCTURES

A type theory for synthetic ∞-categories

Emily Riehl[a] **and Michael Shulman**[b]

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Synthetic $(\infty, 1)$-category theory

Capturing *complete Segal space model*:

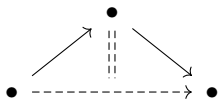
Definition (Synthetic ∞-categories [RS17])

A type A is ...

- **Segal** or a **pre-∞-category** if $\mathrm{res} : A^{\Delta^2} \to A^{\Lambda^2_1}$ is an equivalence.
- **Rezk** or an **∞-category** if it is Segal and $\mathrm{const} : A \to A^{\mathbb{E}}$ is an equivalence.
- **discrete** or an **∞-groupoid** if $\mathrm{res} : A^{\mathbb{I}} \to A$ is an equivalence.

Definition (hom type)

The **hom type** for $a, b : A$ is: $\hom_A(a, b) :\equiv \sum_{f:\mathbb{I}\to A} f(0) = a \times f(1) = b$

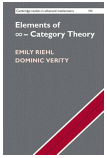


These definitions are mere properties and fully coherent!

Models and other synthetic approaches

- Just as HoTT is the internal language of ∞-toposes $\mathcal{E}$ (Awodey's conjecture solved by [Shu19]), sHoTT is the internal language of simplicial objects $\mathcal{E}^{\Delta^{\mathrm{op}}}$.
- Models will be (internal) complete Segal spaces [Rez01; Ber18], cf. Martini and Wolf's internal ∞-topos theory [MW23].
- Rasekh [Ras25] constructs nonstandard models (as filter quotients).
- There are also close parallels to Riehl–Verity's ∞-cosmos theory [RV22] and Cisinski–Cnossen–Nguyen–Walde's synthetic $(\infty, 1)$-category theory [Cis+25].







Overview

In basic simplicial HoTT, one can develop quite a lot of abstract category theory:

- Basic CT, fibered Yoneda lemma, adjunctions [RS17]
- Limits and colimits [Bar22]
- Cartesian fibrations and generalizations [BW23; Wei24a; Wei24b] (cf. also [RV22])
- Conduché fibrations [Bar24]

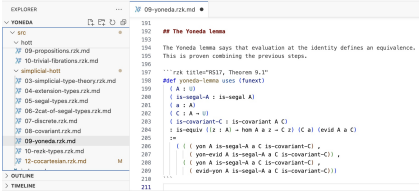
But we want to actually construct nontrivial higher categories. Adding **modal operators** allows for:

- directed univalent universe $\mathbb{S}$ for ∞-groupoids and left fibrations [GWB24] (using *modal operators*), cf. [Rie18; WL20; Wea24]
- functorial Yoneda lemma for presheaves valued in $\mathbb{S}$ [GWB25]
- the synthetic ∞-category Cat of ∞-categories, [GWB26]
- cocompleteness, spectra, and generalized cohomology, *in preparation*
- exponentiability of (co)cartesian fibrations, *in preparation*
- ∞-operads and more higher algebra in Rob Schellingerhout's master thesis

Formalizing ∞-categories in Rzk

- Kudasov has developed the Rzk proof assistant, implementing sHoTT [KSA26]: `https://rzk-lang.github.io/`
- Using Rzk we initiated the first ever formalizations of ∞-category theory.
- In spring 2023, with Kudasov and Riehl we formalized the (discrete fibered) Yoneda lemma of ∞-category theory: `https://emilyriehl.github.io/yoneda/`
- alongside many other results
- Many proofs in this ∞-dimensional setting *easier* than in dimension 1!
- Formalization helped find a mistake in original paper
- More students & researchers joined us developing a library for ∞-category theory: `https://rzk-lang.github.io/sHoTT/`
- **Brand new**: modalities thanks to Islam Talipov. **Join us!**







Agda Synthetic Categories

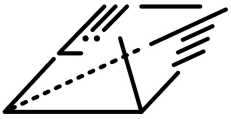
New Agda library by Samuel Toth, Fredrik Bakke, and Mark D. Williams.

Agda Synthetic Categories [index]

2026 · *Samuel Toth with contributions from Fredrik Bakke, Mark Damuni Williams*

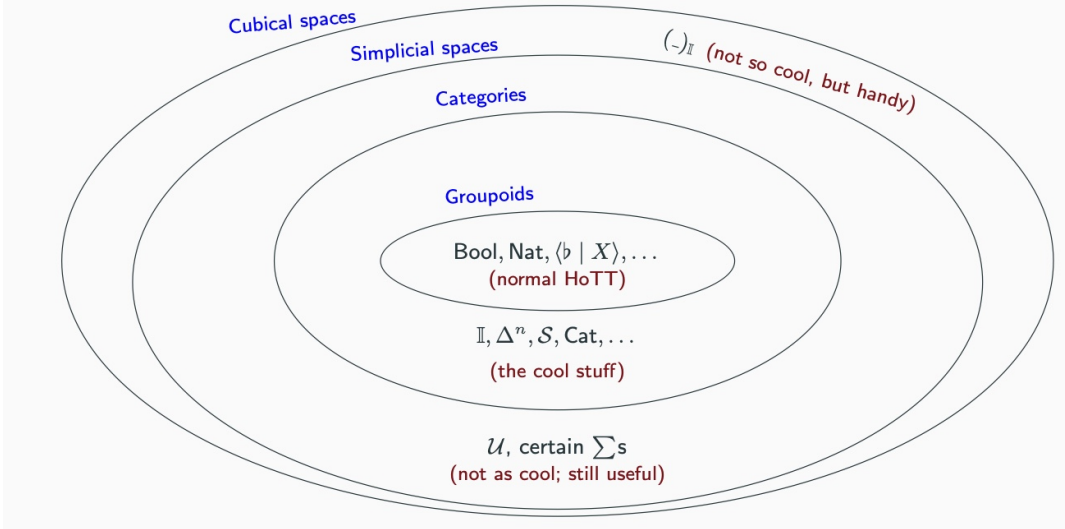
Welcome! This is an online resource and formalisation project written using Jon Sterling's forester tool, containing a mixture of literate agda code and informal mathematics.

- Use `Ctr+k` to search for a specific topic
- Visit our repository to see the source code or get involved
- Read an introduction to the library



- Agda with $\flat \dashv \sharp$
- Fast and thorough progress formalizing triangulated simplicial HoTT
- Towards the universe of spaces and directed univalence
- A lot of progress with new contributors during last week's conference Formalizing Higher Categories at Mittag-Leffler Institute, alongside other frameworks e.g. CaTT [FM17; BFM24], and the ∞-cosmos project in Lean `https://emilyriehl.github.io/infinity-cosmos/`.
- Join in: `https://samtoth.github.io/agda-synthetic-categories/`

Simplicial vs. cubical models



The simplicial submodel

Simplex category $\Delta = \{[n] \mid n \geq 0\}$, **Cube category** $\square = \{[1]^n \mid n \geq 0\}$



Theorem ([KV20; Sat19; SW21])

Simplicial spaces $[\Delta^{\mathrm{op}}, \mathcal{S}]$ *are an essential sub-*∞*-topos of cubical spaces* $[\square^{\mathrm{op}}, \mathcal{S}]$

Internally, a cubical type A is **simplicial** if

$$\mathrm{isSimp}(A) :\equiv \prod_{i,j:\mathbb{I}} \mathrm{isEquiv}(A^{!} : A \to A^{i \leq j \vee j \leq i}).$$

This defines a lex **modality** à la [RSS20]; see also [Wil25].

Theorem ([Sch+26])

Crisp Segal types are simplicial: if $A :_\flat \mathcal{U}$ *is Segal, then* $\mathrm{isSimp}(A)$.

We need the outer cubical layer in order to define the correct (categorical) universes (using [Lic+18; Ril24]).

Triangulated type theory

Triangulated type theory $\mathsf{TT}_{\boxslash}$ is sHoTT + MTT with single mode m and the following mode morphisms:

- **Opposite** op: $\langle \mathrm{op} \mid A \rangle$ has its n-simplices reversed
- **Discretization/core** $\flat$: $\langle \flat \mid A \rangle \to A$ is the maximal subgroupoid of A
- **Codiscretization** $\sharp$: $A \to \langle \sharp \mid A \rangle$ is localization at $\partial\Delta^n \to \Delta^n$ (for crisp closed types)
- **Twisted arrows** tw: $\langle \mathrm{tw} \mid A \rangle$ has as n-simplices:

$$\begin{array}{ccccccccc} a_n & \longleftarrow & \dots & \longleftarrow & a_2 & \longleftarrow & a_1 & \longleftarrow & a_0 \\ \downarrow & & & & & & & & \\ a_{n+1} & \longrightarrow & \dots & \longrightarrow & a_{2n-2} & \longrightarrow & a_{2n-1} & \longrightarrow & a_{2n} \end{array}$$

Mode theory:

$$\flat \circ \flat = \flat \circ \mathrm{op} = \flat \circ \sharp = \mathrm{tw} \circ \flat = \mathrm{op} \circ \flat = \flat \qquad \sharp \circ \sharp = \sharp \circ \flat = \sharp \circ \mathrm{op} = \mathrm{op} \circ \sharp = \sharp$$

$$\mathrm{op} \circ \mathrm{op} = \mathrm{id} \qquad \varepsilon : \flat \to \mathrm{id} \qquad \eta : \mathrm{id} \to \sharp$$

$$\eta \cdot \sharp = \sharp \cdot \eta = \mathrm{id} : \sharp \to \sharp \qquad \flat \cdot \eta = \mathrm{id} : \flat \to \flat$$

plus some coherence conditions for tw

Axioms for triangulated type theory

- distributive bipointed interval $\mathbb{I}$
- homotopy theory and modalities interact well
- For cubical types, $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

Axiom (Synthetic quasi-coherence (SQC) [Ble23])

Let $\mathbb{I} \to A$ be a finitely presented $\mathbb{I}$-algebra, i.e., $A \simeq \mathbb{I}[x_1, \ldots, x_n]/(r_1 = s_1, \ldots, r_n = s_n)$, then the evaluation map is an equivalence:

$$\mathrm{ev} \equiv \lambda a, f.f(a) : A \xrightarrow{\simeq} (\hom_{\mathbb{I}}(A, \mathbb{I}) \to \mathbb{I})$$

Versions of the latter axiom appear in synthetic differential geometry (Kock–Lawvere axioms) [Koc77], synthetic algebraic geometry [CCH24], and synthetic domain theory [SY25]. For more context and a new categorical discussion see [Mye25].

Lemma (Phoa's principle [Pho90] and [PS25])

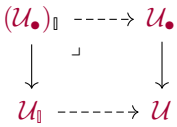
$(\mathbb{I} \to \mathbb{I}) \simeq \Delta^2$

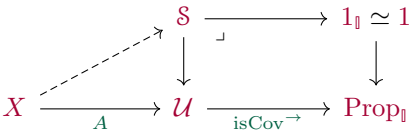
In particular, Δ^n is a category.

The universe of spaces

Let $\mathrm{isCov} : \mathcal{U}^X \to \mathrm{Prop}$ be the predicate expressing that a family is right orthogonal to the inclusion $\{0\} \hookrightarrow \mathbb{I}$, i.e. a discrete *covariant family*.
Using the amazing right adjoint $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ we define the **universe of spaces**:

$$\frac{\mathrm{isCov}_{\to} : \mathcal{U}^{\mathbb{I}} \to \mathrm{Prop}}{\mathrm{isCov}^{\to} : \mathcal{U} \to \mathrm{Prop}_{\mathbb{I}}}$$





Theorem

$\mathcal{S}$ is a category (Segal and Rezk), has as points the groupoids, is closed under Σ, identity types, and finite (co)limits.
It classifies amazingly covariant families in that

$$\langle \flat \mid X \to \mathcal{S} \rangle \simeq \langle \flat \mid \sum_{A : X \to \mathcal{U}} \mathrm{isCov}(A) \rangle.$$

Directed univalence

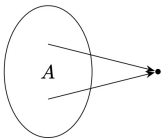
1. Since $\mathbb{S}$ classifies (amazingly) covariant families, there is a map

$$\mathrm{arrtofun} :\equiv \lambda F.(F\,0, F\,1, \alpha_!^F : F\,0 \to F\,1) : (\Delta^1 \to \mathcal{S}) \to \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big).$$

2. In the other direction, we consider the **mapping cone/directed glue type** (cf. cubical type theory and Weaver–Licata '20):

$$\mathrm{Gl} :\equiv \lambda A, B, f.\lambda i. \sum_{b:B} (i = 0) \to f^{-1}(b) : \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big) \to (\Delta^1 \to \mathcal{S})$$

3. Then argue that these are inverses up to homotopy.



Application: directed homomorphism principle (DHP)

Theorem (DHP for pointed spaces)

Let $\mathcal{S}_* :\equiv \sum_{A:\mathcal{S}} A$. *Then for* $(A, a), (B, b) : \mathcal{S}_*$ *we have:*

$$\mathrm{hom}_{\mathcal{S}_*}((A, a), (B, b)) \simeq \sum_{f:A\to B} f(a) = b$$

Theorem (DHP for monoids)

Consider the type (category!) of (set-)monoids

$$\mathrm{Monoid} :\equiv \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\varepsilon:A} \sum_{\cdot:A\times A} \mathrm{isAssoc}(\cdot) \times \mathrm{isUnit}(\cdot, \varepsilon).$$

Then homomorphisms from $(A, \varepsilon_A, \cdot_A, \alpha_A, \mu_A)$ *to* $(B, \varepsilon_B, \cdot_B, \alpha_B, \mu_B)$ *correspond to set maps* $A \to B$ *compatible with multiplication and units.*

We also get higher versions!

Functorial Yoneda lemma

- So far: $\mathrm{hom}_C : C \times C \to \mathcal{U}$
- We have a *fibered Yoneda lemma* [RS17; BW23; KRW04].
- **Problem:** How to get the right hom-bifunctor $\langle \mathrm{op} \mid C \rangle \times C \to \mathcal{S}$, with $\mathcal{S}$ the ∞-category of ∞-groupoids by [GWB24]?
- **Solution:** Use the twisted arrow fibration $\langle \mathrm{tw} \mid C \rangle \to \langle \mathrm{op} \mid C \rangle \times C$.
- Then, from $\Phi_C : \langle \mathrm{op} \mid C \rangle \times C \to \mathcal{S}$ define the **Yoneda embedding** $\mathbf{y}_C := \lambda c.\Phi_C(-, c) : C \to \widehat{C} := \mathcal{S}^{\langle \mathrm{op} \mid C \rangle}$.

Theorem (Functorial Yoneda lemma)

There is a natural isomorphism $\Phi(\mathbf{y}(-), -) \cong \mathrm{eval} : \langle \mathrm{op} \mid C \rangle \times \widehat{C} \to \mathcal{S}$.

More category theory in triangulated type theory

Already $\mathcal{S}$ unlocks a lot of nice results: functorial Yoneda lemma, Kan extensions, (co)final functors and Quillen's Theorem A, Smooth and proper functors

The ∞-category of ∞-categories

- Recently, we constructed the long desired **∞-category of ∞-categories** Cat synthetically.
- Just as $\mathbb{S}$ is the classifier for (amazingly) covariant fibrations, Cat should be the classifier for (amazingly) **cocartesian fibrations**, see [BW23].
- A direct attempt to define Cat in this way seems intractible. However, an approach via **coherently locally cocartesian arrows** works. Over $\mathbb{I}$ this amounts to:

$$\mathrm{isLCC} : \prod_{A:\mathcal{U}^{\mathbb{I}}} \prod_{a:(i:\mathbb{I})\to A(i)} \mathrm{Prop}$$

$$\mathrm{isLCC}(A,a) :\equiv \prod_{b:(i:\mathbb{I})\to A(i)} \prod_{p:a0=b0} \mathrm{isContr}\left(\sum_{t:(i,j:\Delta^2)\to A(i)} t|_{\Lambda^2_0} = [a,b,p] \right)$$

- We then define Cat by:

$$\mathrm{Cat} :\equiv \sum_{A:\mathcal{U}_{\not\square}} \mathrm{isRezk}(A) \times \mathrm{aIsInner}(A) \times \mathrm{aHasLCCLifts}(A)$$

Rob Schellingerhout observed that coherently locally cocartesian fibrations are already cocartesian.

Cocartesian directed gluing

Definition

Let $F_0, F_1 :\flat\ X \to \mathcal{U}_{\not\square}$ be cocartesian fibrations, and $\alpha :\flat\ \prod_{x:X} F_0(x) \to F_1(x)$ be a cocartesian functor. The **directed gluing** is given by:

$$\mathrm{Gl}(F_0, F_1, \alpha) : \mathbb{I} \times X \to \mathcal{U}_{\not\square}$$
$$\mathrm{Gl}(F_0, F_1, \alpha)(i, x) :\equiv \sum_{f:F_1(x)} (i = 0) \to \alpha(x)^{-1}(f)$$

Theorem

If X is a category and F_0, F_1, α are as above, then $\mathrm{Gl}(F_0, F_1, \alpha)$ is cocartesian.

Classifying property and directed univalence

Theorem ([GWB26])

1. Cat *classifies (amazingly) cocartesian families in* $\mathcal{U}_{\square}$*:*

$$\begin{array}{ccc} E \simeq B \times_{\mathrm{Cat}} \mathrm{Cat}_* & \longrightarrow & \mathrm{Cat}_* \\ \downarrow{\scriptstyle \xi} & \lrcorner & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\chi_\xi} & \mathrm{Cat} \end{array}$$

2. Cat *is* ***directed univalent****: For* $A, B :_\flat \mathrm{Cat}$*, we have*

$$\mathrm{dua} \ : \mathrm{hom}_{\mathrm{Cat}}(A, B) \simeq \langle \flat \mid A \to B \rangle$$

3. Cat *is a category, i.e., Segal, Rezk, and simplicial.*

Applications

- We get an inclusion $i : \mathcal{S} \hookrightarrow \mathrm{Cat}$ with both left and right adjoint.
- We can define full subcategories $C_\varphi \to C$ for propositions $\varphi :_\sharp \langle \flat \mid C \rangle \to \mathrm{Prop}_\square$.
- In particular, we can define the categories MCat of symmetric monoidal ∞-categories, E_1-monoids Mon, and ∞-groups as full subcategories of $\mathcal{S}^{\langle \mathrm{op} \mid \Delta \rangle}$.
- Then can define K-theory of a monoidal 1-category as the composite

$$\mathrm{MCat}_1 \to \mathrm{MCat} \to \mathrm{Mon} \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*.$$

  n-truncation yields the n-th K-group.
- We can exhibit $\sum_{A:\mathrm{Cat}} A^C$ as the lax slice for $C :_\flat \mathrm{Cat}$.

Outlook

- more higher algebra
- internal higher topos theory
- metatheory of (higher?) type theories internally in type theory
- computational (cubical) version and metatheory of modal sHoTT
- Connections to other frameworks [Cis+25; SW26; RKB24; dEs23; KS25]
- Connections to synthetic higher & differential geometry [Sch13; SS12; Shu18; Wel18; CCH24; MR23] and synthetic Stone duality & (light) condensed type theory ([BC24; Che+24; CGM25]), synthetic higher domain theory [PS25; SY25; Bak+26], higher logical relations and computability
- more formalization in Rzk, Agda Synthetic Categories, ...
- Higher Observational Simplicial Type Theory? [Shu+25]
- Synthetic (∞, n)-categories using Θ [LM26]
- ...

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