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From Differential Linear Logic to Functional Analysis, and back

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Objectives

- A quick introduction to Linear Logic (LL), Differential Linear Logic (DiLL) and their **models**.
- Most importantly, I want to give you a taste of how denotational semantics might be used in the context of linear logic: **semantics** is used to **enrich the syntax**.
- We will see how the theory of **topological vector spaces** and **distributions** is relevant to enrich Differential Linear Logic.

Linear Logic

Without structural rules

Deleting or restricting structural rules has a huge impact on the other connectives:

$$\frac{\Gamma \vdash A, B}{\Gamma \vdash A \wedge B} \text{ Multiplicative R-}\vee \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \text{ Multiplicative R-}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \text{ Additive R-}\vee\ 1 \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \vee B} \text{ Additive R-}\wedge$$

Additive and multiplicative rules are no longer equivalent. Hence the introduction of two disjunctions and two conjunctions:

	$\vee$	$\wedge$
Additive	$\oplus$	$\&$
Multiplicative	$\parr$	$\otimes$

Structure first

Linear Logic (LL) restricts the structural rules of Intuitionistic Logic to exponential connective:

Structural rules in LL

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\text{ contraction} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\text{ weakening}$$

*For a proof of B under the hypothesis $!A$, I might have used $!A$ several times (**contraction**), or none (**weakening**).*

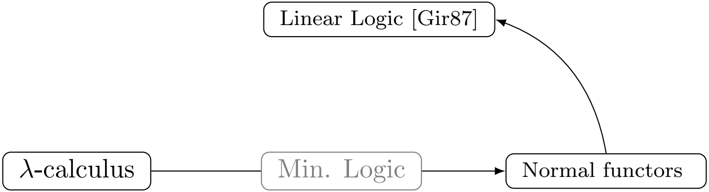
$!A$ can be interpreted as a collection of resources of type A.

What's missing ?

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\text{ dereliction} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\text{ promotion}$$

In the beginning was denotational semantics

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



Normal functors and resources

not so easy, and uses all the preservation properties. Now, by counting the normal forms, it is possible to express normal functors (up to isomorphism) by power series: in other terms normal functors are analytic (and conversely). If we consider the normal functors from $\mathbf{SET}^A$, to $\mathbf{SET}^B$, then by looking at the coefficients of their power series expansion, we see that such functors can be viewed as the objects of a new category $\mathbf{SET}^{\mathrm{Int}(A)\cdot B}$. All usual operations (typically: λ-abstraction, application of a function to an argument) can be represented by means of normal functors, i.e., power series. It may be of interest to say a word about the meaning of a basic monomial, for instance $m_a^2 \cdot m_b^3$: such a monomial contributes to some multiplicity, and is expressed from the multiplicities m_a and m_b; concretely this means that the input 'a' has been used twice while the input 'b' has been used three times . . . In particular, linear

functors are those where informations are used once; for instance application of a function to an argument is linear in the function (not at all in the argument!), similarly projections are linear. It seems that the general idea of an 'evaluation operation' is well represented by the idea of a linear functor: it is by following this

Linear Logic, vocabulary

Implications are linear, hence the introduction of a linear arrow:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

The usual implication of LJ is retrieved thanks to a notation :

$$A \Rightarrow B := !A \multimap B$$

A proof of $A \vdash B$ is linear wrt A. A proof of $!A \vdash B$ is non-linear wrt A.

- A linear proof is in particular non-linear,
- A linear proof will make use only once of its hypothesis A,
- A non-linear proof will use its hypothesis A an arbitrary number of times.

Classical Linear Logic

I'll be considering only classical linear logic, where sequents are monolateral by default but can be made bilateral for better intuitions.

$$\Gamma \vdash A \qquad \vdash \Gamma^{\perp}, A$$

Formulas for Linear Logic

$$A, B := X \mid X^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \mid\, !A \mid\, ?A$$

where $\otimes$ (resp $\&$) denotes the multiplicative (resp. additive) conjunction and $\parr$ (resp $\oplus$) denotes the multiplicative (resp. additive) disjunction.

An involutive linear negation

$$\begin{array}{rclrcl}
(A \,\&\, B)^{\perp} & = & A^{\perp} \oplus B^{\perp} & (A \oplus B)^{\perp} & = & A^{\perp} \,\&\, B^{\perp} \\
(A \parr B)^{\perp} & = & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & = & A^{\perp} \parr B^{\perp} \\
!A^{\perp} & = & ?A^{\perp} & ?A^{\perp} & = & !A^{\perp}
\end{array}$$

Monolateral Classical Linear Logic

$$\dfrac{}{\vdash A, A^\perp}\ \text{axiom} \qquad \dfrac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\dfrac{}{\vdash 1}\ (1) \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\dfrac{}{\vdash \Gamma, \top}\ \top \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \qquad \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \qquad \dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad \dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

$$\Gamma \vdash \Delta \qquad A_1 \otimes \cdots \otimes A_n \vdash B_1 \parr \cdots \parr B_n$$

Linear and non-linear implication

$$A \multimap B := A^\perp \parr B \qquad A^\perp \equiv A \multimap \perp$$

$$A \Rightarrow B := !A \multimap B$$

Intuitionistic Linear Logic

Formulas for ILL

$$A, B := X \mid 0 \mid 1 \mid \top \mid A \otimes B \mid A \multimap B \mid A \oplus B \mid A \,\&\, B \mid !A$$

Deduction rules for ILL

$$\frac{}{A \vdash A}\ \text{ax} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{cut} \qquad \frac{}{\vdash 1}\ 1\text{R} \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\ 1\text{L}$$

$$\frac{\Gamma, \vdash A \qquad \Delta, \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\text{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\text{L} \qquad \frac{\Gamma, A \vdash B}{\vdash \Gamma, A \multimap B}\ \multimap\text{R}$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta, A \multimap B \vdash C}\ \multimap\text{L} \qquad \frac{}{\Gamma, \vdash \top}\ \top \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \&\text{R}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma A_1 \,\&\, A_2 \vdash C}\ \&\ L_i \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}\ \oplus\ R_i \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus\text{L}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \mathsf{p}$$

A fruitful methodology

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
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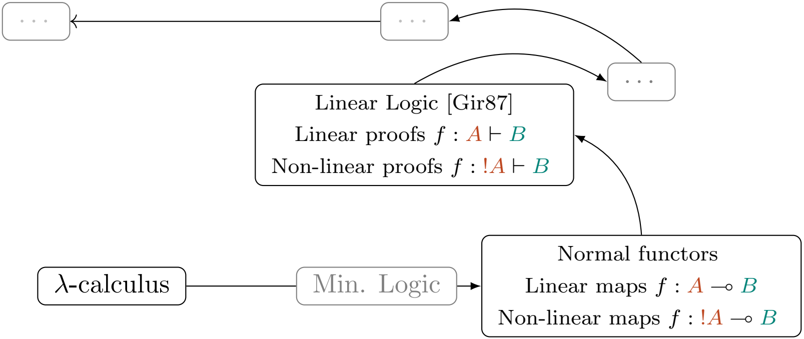

The Relational Model

An important model for Linear Logic and its variants, that expresses well the notion of resources and linear argument at stakes.

- Formulas are interpreted by sets $\{a_1, \dots, a_n\}$,
- Proofs are interpreted relations $[\![A \vdash B]\!] = R \subseteq A \times B$,
- $!A := \mathfrak{M}_f(A)$ is the set of *finite multi-sets* of A,

$$\{a_1^{\alpha_1}, \cdots, a_n^{\alpha_n}\}$$

Multiplicative Connectives

- $\otimes$ and $⅋$ are interpreted as the cartesian product of sets, with $[\![1]\!] = [\![\bot]\!] = \{*\}$.

Additive Connectives

- $\&$ and $\oplus$ are interpreted as the disjoint unions of sets, with $[\![0]\!] = [\![\top]\!] = \emptyset$.

Categorical models of Linear Logic

Definition A categorical model of Intuitionnistic Linear Logic consists of a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$ making it monoidal closed, a cartesian product $(\&, 0)$ with a monoidal co-monad $! : \mathcal{L} \to \mathcal{L}$:

Seely isomorphisms

$$m_{A,B} : !(A \,\&\, B) \to !A \otimes !B \qquad m_0 : !0 \to 1$$

- From the co-monad rule, one gets dereliction and promotion:

$$\mathsf{d}_A : !A \to A \qquad \mathsf{p}_A : !A \to !!A$$

- From the monoidality of ! and the cartesian product, one gets w and d.
- An equivalent monoidality requirement :

$$\Phi : !A \otimes !B \xrightarrow{m^{-1}} !(A \times B) \xrightarrow{\mathsf{p}} !!(A \,\&\, B) \xrightarrow{m} !(!A \otimes !B)) \xrightarrow{!(\mathsf{d} \otimes \mathsf{d})} !(A \otimes B)$$

- Classical LL : a $*$-autonomous structure on top of it.

Categorical Semantics of Linear Logic, Paul-André Melliès, 2009

The Relational Model

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$$\{a_1^{\alpha_1}, \cdots, a_n^{\alpha_n}\}$$

Exponential rules

- $\mathsf{d}_A^\perp = \{(\{a\}, a) | a \in A\}$
- $\mathsf{c}_A^\perp = \{(m_1 \cup m_2, (m_1, m_2) | m_1, m_2 \in \mathfrak{M}_f(A)\}$
- $\mathsf{w}_A^\perp = \{(\emptyset, *)\}$
- $\mathsf{p}_A^\perp = \{(m_1 \cup \cdots \cup m_n, [m_1, ..., m_n]) \mid n \in \mathbb{N}, m_i \in !A\}$

Variants of Linear Logic

Polarized Linear Logic

	Conjunction	Disjunction
Invertible rule = negative connective	$\&$	$⅋$
Non-invertible = positive connective	$\otimes$	$\oplus$

$$\text{Negative Formulas: } N, M := a \mid ?P \mid N ⅋ M \mid \bot \mid N \& M \mid \top \mid \uparrow P$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1 \mid \downarrow N$$

This leads to $\mathrm{LL_{pol}}$ with focused sequents, where a positive formula is isolated:

$$\vdash N_1, ..., N_n \mid P.$$

Chiralities by Melliès are categorical models of polarized MALL (LL with only $\otimes$ and $⅋$).

$$\begin{tikzcd} (\mathscr{P}^{op}, \otimes) \arrow[r, bend left, "(\_)^{\perp_L}"] \arrow[r, phantom, "\perp"] & (\mathscr{N}, ⅋) \arrow[l, bend left, "(\_)^{\perp_R}"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{N} \arrow[l, bend left, "\downarrow"] \end{tikzcd}$$

along with a suitably natural family of bijections relaxing monoidal closedness:

$$\chi_{p,n,m} : \mathscr{N}(\uparrow p, n ⅋ m) \simeq \mathscr{N}(\uparrow (p \otimes n^{\perp_N}), m)$$

Variants of Linear Logic

Graded Linear Logic $B_{\mathcal{S}}LL$

Graded Linear Logic expresses finely the notion of resources. The exponentials are now indexed by elements of a semiring.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\,\mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A}\,\mathsf{d} \qquad \frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\,\mathsf{d}_I$$

$$\frac{\vdash ?_Y \Gamma, A}{\vdash ?_{x \times Y} \Gamma, !_x A}\,\mathsf{p}$$

Semantics :

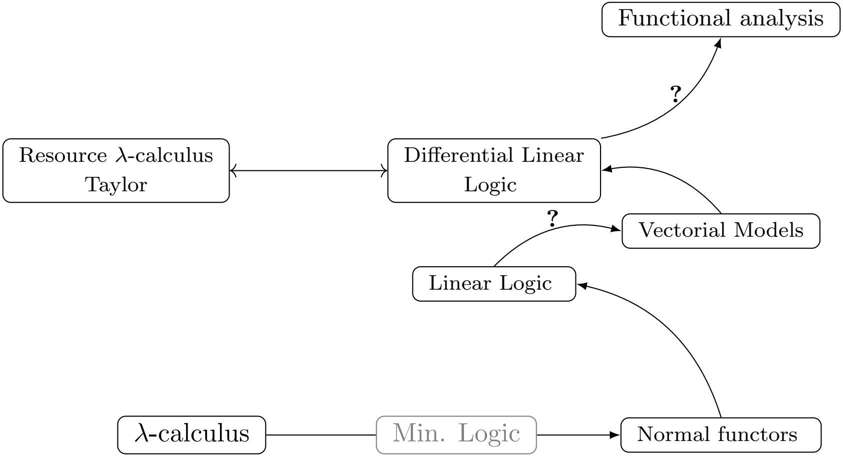
- $[\![!A \multimap V]\!] = \{f = \sum_{k \in \mathbb{N}} a_k x^k\}$
- $[\![!_n A \multimap V]\!] = \{f = \sum_{j \leq n} a_k x^k\}$

Graded co-monads interpret the exponential.

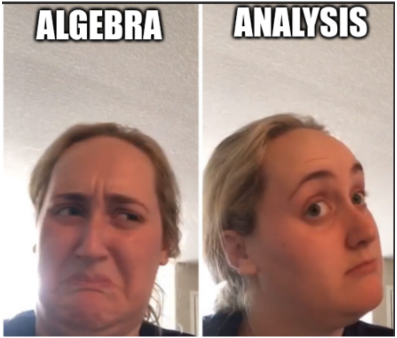
Jean-Yves Girard, Andre Scedrov, Philip J. Scott, 1992

Martin Hoffman, Ugo Dal Lago,2009

Programs **Logic** **Semantics**



Differential Linear Logic



How far can we push the Linearity analogy ?

- Let's make Rel a vectorial model

One stumbles on so-called Köthe spaces, spaces of sequences studied independently by mathematicians:

$$\text{For } E \subset \mathbb{R}^{\mathbb{N}}\text{: } E^{\perp} := \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid \forall \lambda \in E, \sum_n |\lambda_n \alpha_n| < \infty\}.$$

Definition

- A **perfect sequence space** is the data (X, E_X) of a subset $X \subset \mathbb{N}$ and $E_X \subset \mathbb{K}^X$ such that $E_X^{\perp\perp} = E_X$.
- The space $E \multimap F$ of **linear continuous maps** from E_X to F_Y correspond to the subset $\mathbb{K}^{X \times Y}$ of all M such that the sum:

$$\sum_{i,j} M_{i,j} x_i y'_j$$

is absolutely converging for all $x \in E$ and $y' \in F^{\perp}$.

T. Ehrhard. On Köthe sequence spaces and linear logic. MSCS, 2002.

Differentiable maps in Köthe spaces

Exponents. If μ is a finite multiset of X and $x \in E$, we write:

$$x^{\mu} = \prod_{n} x_n^{\mu(n)}.$$

Power series We define the set of scalar entire maps $E \Rightarrow \mathbb{K}$ as the vector space of matrices $M \in \mathbb{K}^{\mathcal{M}_f(X)}$ such that for all $x \in E$, the following sum converges absolutely:

$$f(t) = \sum_{\mu \in \mathcal{M}_f(X)} M_{\mu} t^{\mu}$$

Power series are differentiable !

$$f(t) = \sum_{\mu \in \mathcal{M}_f(X)} M_{\mu} t^{\mu} \rightsquigarrow D_0 f(t) = \sum_{x \in X} a_{\{t\}} t$$

Köthe spaces are a particular example of **quantitative models**.

From linearity to quantitative models

Functions	**Programs**
Power series	Resources consumption or Probabilistic sums
$f = \sum_n f_n$	$p(x) = \sum p_n$

From linearity to quantitative models

Functions	**Programs**
Power series $f = \sum_n f_n$	Resources consumption or Probabilistic sums $p(x) = \sum p_n$
f_n is n-linear	p_n consumes exactly n-times its resources.
f is *Taylor* $f = \sum_n \frac{1}{n!} D_0^{(n)} f$	Programs can be approximated $(M)S = \sum_n \frac{1}{n!} < M > S^{\otimes^n}$

- Experimentally, quantitative semantics is what gets you (AxF)
- It leads to new proof techniques on λ-calculus.

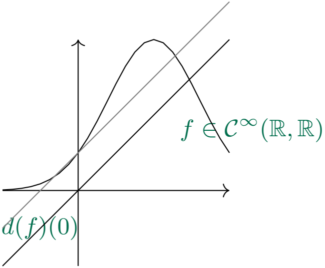
Work by Accatolli, Crubillé, Ehrhard, Guerrieri, Kesner, Manzonetto, Ong, Pagani, Tasson, Vaux

Even when we really try to avoid it, we stumble back on quantitative constructions [Dabrowski, K. 2018]

Core intuition: Differentials are enough to compute

Differentiation

- Differentiation is finding the best linear approximation to a function at a point.



For $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$
$D_x f : v \in \mathbb{R} \mapsto f'(x) \cdot v \in \mathbb{R}$

For $f : E \to F$, $x \in E$
$D_x f : v \in E \mapsto D_x f(v) \in F$

$$\text{Chain Rule}: \; D_0(g \circ f) = D_{f(0)}(g) \circ D_0(f)$$

- Differentiation is a mathematical operation at the core of numerical computations, itself the reason for computer to be developed. However, it still needs to be tailored for computer science.

A symmetric to the dereliction

Remember:

$$\llbracket A \multimap B \rrbracket = \{\ell : A \to B, f \text{ is linear}\}$$
$$\llbracket !A \multimap B \rrbracket = \{f : A \to B, f \text{ power series, smooth, polynomial, } \dots\}$$

The dereliction rule allows one to go from a linear map to a non-linear one:

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

To be able to differentiate proofs, one needs to introduce a co-dereliction :

$$\frac{f : !A \vdash B}{\mathrm{D}_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Adding the co-dereliction to the exponential group

Exponential law of LL + co-dereliction:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathsf{w} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,\mathsf{p} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{\mathsf{d}}$$

How to express the cut-elimination procedure ? The key is to look at the semantics !

Example : dereliction and co-dereliction

$$\frac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\,\bar{\mathsf{d}} \qquad \dfrac{\ell : A \vdash B}{\ell : !A \vdash B}\,\mathsf{d}}{\vdash \Gamma, \Delta}\,\mathrm{cut}$$

$$\rightsquigarrow \qquad \frac{\vdash \Gamma, v : A \qquad \ell :\vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(v) = \ell(v)}\,\mathrm{cut}$$

To express a cut-elimination rule between p *and* $\bar{\mathsf{d}}$*, one needs to add other rules ! Let's understand why by exploring a the semantics a bit more.*

Denotational Intuitions for "Why Not" "?"

In classical Linear Logic

$$[\![?A]\!] \simeq [\![A^\perp \Rightarrow \perp]\!]$$

Let's assume, as in most models of Differential Linear Logic:

$$[\![\perp]\!] = \mathbb{R} \qquad [\![A \Rightarrow B]\!] = \mathcal{C}^\infty([\![A]\!], [\![B]\!])$$

Then

$$[\![?A]\!] \subseteq \{f | f \in \mathcal{C}^\infty([\![A^\perp]\!], \mathbb{R})\}$$

This gives us another interpretation for exponential rules of Linear Logic in terms of basic calculus rules.

Denotational Intuitions for exponential rules

Weakening and contraction

$$\frac{c :\vdash \Gamma}{cst_c : !A \vdash \Gamma}\, w$$

The constant function is non-linear

$$\frac{x : !A, y : !A \vdash g(x, y) : \Gamma}{x : !A \vdash g(x, x) : \Gamma}\, \mathsf{c}$$

Identification of variables

Dereliction and promotion

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\, \mathsf{d}$$

linear $\rightsquigarrow$ *non-linear*

$$\frac{\mathrm{f}{:}!\Gamma \vdash A}{(x \mapsto (g \mapsto g(f(x)) : !\Gamma \vdash !A}\, \mathsf{p}$$

Non-linear composition

Denotational Intuitions for exponential rules

Weakening and contraction

$$\frac{\vdash \Gamma}{\vdash \Gamma, (cst_1 : a' \mapsto 1) : ?A}\, w$$

The constant function is non-linear

$$\frac{\vdash \Gamma, f : ?A, f : ?A}{\vdash \Gamma, f \cdot g : ?A}\, \mathsf{c}$$

The multiplication of scalar functions

Dereliction and promotion

$$\frac{\vdash \Gamma, v : A}{\vdash \Gamma, ((x : A^{\perp}) \mapsto x(v)) : ?A}\, \mathsf{d}$$

linear $\rightsquigarrow$ *non-linear*

$$\frac{\vdash ?\Gamma, v : A}{\vdash ?\Gamma, \delta_v : !A}\, \mathsf{p}$$

Diracs

Distributions instead of resources intuitions for !

$$[\![!A]\!] \simeq [\![(!A \multimap \perp) \multimap \perp]\!] \simeq [\![(A \to)^{\perp}]\!]$$

Let's assume, as in most models of Differential Linear Logic:

$$[\![\perp]\!] = \mathbb{R} \qquad [\![A \Rightarrow B]\!] = \mathcal{C}^{\infty}([\![A]\!], [\![B]\!])$$

$$[\![A^{\perp}]\!] = [\![A]\!]' = \mathcal{L}(A, \mathbb{R}) \qquad \text{Linear Negation is interpreted by Dual}$$

Then the ! is interpreted as a space of distributions with compact support:

$$[\![!A]\!] \subseteq \mathcal{C}^{\infty}([\![A]\!], \mathbb{R})'$$

- Distributions are linear scalar maps acting on some smooth functions.
- e.g . : $\delta_x : f \mapsto f(x)$, or if g has compact support $T_g : f \mapsto \int f(t)g(t)dt$

Convolution, the monoidal operation on distributions:

$$\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y))) \qquad \delta_x * \delta_y = \delta_{x+y}$$

Unchained melody

Cut-elimination between p and $\bar{\mathsf{d}}$:

$$\dfrac{\dfrac{!A \vdash B}{!A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash A}{\Gamma \vdash !A}\,\bar{\mathsf{d}}}{A \vdash B}\,\mathrm{cut} \qquad\qquad \rightsquigarrow \qquad\qquad \cdots$$

Unchained melody

Cut-elimination between p and $\overline{\mathsf{d}}$:

$$\dfrac{\dfrac{(f : x : !A \vdash f(x)) : B}{(\delta_f : x \mapsto \delta_{f(x)}) : !A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\overline{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) : !B}\,\mathrm{cut} \qquad \rightsquigarrow \qquad \cdots$$

Unchained melody

Cut-elimination between p and $\bar{\mathsf{d}}$:

$$\dfrac{\dfrac{(f : x : !A \vdash f(x)) : B}{(\delta_f : x \mapsto \delta_{f(x)}) : !A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) : !B}\,\mathrm{cut} \qquad \rightsquigarrow \qquad \cdots$$

$$\begin{aligned} D_0(\delta_f)(v) &= (g \mapsto D_0(g \circ f)(v)) \\ &= (g \mapsto D_{f(0)}(g)(D_0(f)(v)) \text{ thanks to the chain rule} \end{aligned}$$

How to translate $D_{f(0)}(D_0(f)(v))$ as a proof tree? One needs two rules:

- One to interpret $(f \mapsto f(0)) : !A$, the co-weakening
- One to interpret $(D_0(\_)(v) \otimes (f \mapsto f(0)) \mapsto D_{f(0)}(\_)(v)) : !A \otimes !A \to !A)$, the co-contraction

Differential Linear Logic

DiLL adds to LL the co-dereliction, co-contraction, co-promotion rules:

Exponential rules of DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

$$\frac{}{\vdash !A}\ \overline{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \overline{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \overline{\mathsf{d}}$$

Denotational intuitions for co-structural rules of DiLL

Co-weakening

Computing the value at 0 of a smooth map f

$$\frac{}{\vdash \delta_0 : !A}\ \overline{\mathsf{w}} \qquad\qquad \frac{f\colon\ !A \vdash B}{\vdash f(0) : B}\ \overline{\mathsf{w}}$$

Co-contraction

Summing in the domain of smooth maps

$$\frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{c} \qquad\qquad \frac{f\colon\ !A, \vdash B}{(x,y) \mapsto f(x+y)\colon\ !A, !A \vdash !B}\ \bar{\mathsf{c}}$$

Cut-elimination between promotion and co-dereliction

$$\dfrac{\dfrac{f : (x : !A \vdash f(x)) : B}{\delta_f : x \mapsto \delta_{f(x)} : !A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) : !B}\,\mathrm{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{(x : !A \vdash f(x)) : B \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(f)(v) : B}\,\mathrm{cut}}{\Gamma \vdash D_0(\_)(D_0(\_)(v)) : !B}\,\bar{\mathsf{d}} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \delta_0 : !A}\,\bar{\mathsf{w}} \qquad (x : !A \vdash f(x)) : B}{\vdash f(0)) : B}\,c}{\vdash \delta_{f(0)} : !B}\,\mathsf{p}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) = \delta_{f(0)} * \mathrm{D}_{(\_)}(D_0(f)(v)) : !B}\,\bar{\mathsf{c}}$$

The chain-rule with contexts:

$$id_{!A} \otimes \bar{\mathsf{d}}_A ; \bar{\mathsf{c}}_A ; \mathsf{p}_A = \mathsf{c}_A \otimes \bar{\mathsf{d}}_A ; id_{!A} \otimes \bar{\mathsf{c}}_A ; \mathsf{p}_A \otimes \bar{\mathsf{d}}_{!A} ; \bar{\mathsf{c}}_{!A}$$

Cut-elimination in a nutshell

Cut-elimination in DiLL is symmetric

- Cut-elimination of co-structural rules between them are alike cut-eliminations of structural rules.
- Cut-elimination between structural and co-structural rules are alike.

Categorically speaking, if we consider the natural transformations interpreting the derivation rules:

- $\overline{\mathsf{d}}; w = 0$ and $\overline{\mathsf{w}}; \mathsf{d} = 0$
- $\overline{\mathsf{w}}; \mathsf{w} = id$ and $\overline{\mathsf{d}}; \mathsf{d} = id$
- $\overline{\mathsf{c}}; \mathsf{w} = \mathsf{w} \otimes \mathsf{w}$ and $\overline{\mathsf{w}}; \mathsf{c} = \overline{\mathsf{w}} \otimes \overline{\mathsf{w}}$
- $\mathsf{c}; \overline{\mathsf{d}} = \overline{\mathsf{w}} \otimes \overline{\mathsf{d}} + \overline{\mathsf{d}} \otimes \mathsf{w}$ and $\mathsf{d}; \overline{\mathsf{c}} = \mathsf{w} \otimes \mathsf{d} + \mathsf{d} \otimes \mathsf{w}$
- $(!, \mathsf{w}, \mathsf{c}, \overline{\mathsf{w}}, \overline{\mathsf{c}})$ is a commutative bialgebra

This can be made more synthetic in Categorical Models of DiLL.

Categorical models of DiLL

There's a rich literature on differential categories by Blute, Cockett, Scott, Pacaud-Lemay ...

- A $*$-autonomous category $(\mathcal{L}, \otimes, \mathbb{R})$
- with **an additive structure** (ie finite product = finite coproduct, meaning the category is enriched over commutative monoids),
- A co-monad $(!, d, p)$ *strongly* monoidal $(\mathcal{L}_!, \times, 0) \to (\mathcal{L}, \otimes, 1)$
- A **differentiation** $\bar{\mathsf{d}} : Id \to\, !$ such that $\bar{\mathsf{d}}; d = Id$ and $\bar{\mathsf{d}}; p$ is the chain rule:

$$id_{!A} \otimes \bar{\mathsf{d}}_A; \bar{\mathsf{c}}_A; \mathsf{p}_A = \mathsf{c}_A \otimes \bar{\mathsf{d}}_A; id_{!A} \otimes \bar{\mathsf{c}}_A; \mathsf{p}_A \otimes \bar{\mathsf{d}}_{!A}; \bar{\mathsf{c}}_{!A}$$

$\bar{\mathsf{c}}$ and $\bar{\mathsf{w}}$ are constructed from the algebra structure on each object as follows, dually to c and w.

There are other axiomatizations, in particular one can ask for a functor $\mathrm{D} : I \downarrow \mathcal{C}^\infty \to \mathcal{L}$.

"Differential Categories Revisited" Rick Blute, Robin Cockett, J-S. Pacaud Lemay, R.A.G. Seely", 2020

How far can we push the differentiation analogy ?

Looking for:

a differential category within mainstream analysis ?

- A cartesian closed category with smooth functions as morphisms
- A monoidal closed category of topological vector spaces
- A $*$-autonomous category .

How far can we push the differentiation analogy ?

Looking for:

a differential category within mainstream analysis ?

- A cartesian closed category with smooth functions as morphisms
  complete spaces
- A monoidal closed category of topological vector spaces
  good behaviour of topological tensor products
- A $*$-autonomous category
  reflexive spaces.

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

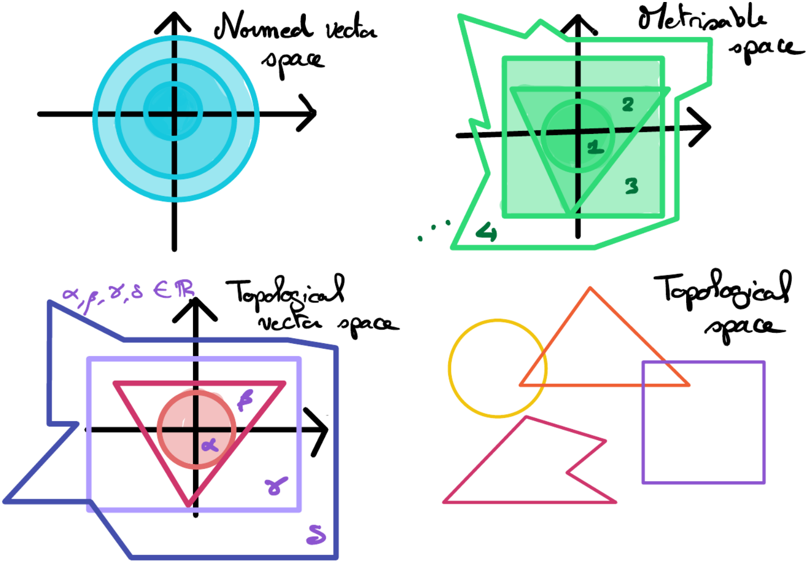
- Fails on analytic functions [Girard99].
- More general space of functions are metrizable but not normed.

We can't restrict ourselves to normed spaces.

We work with Hausdorff topological vector spaces: real or complex vector spaces endowed with a Hausdorff topology making addition and scalar multiplication continuous.

Two layers: algebraic and topological constructions

A few intuitions on topological vector spaces



Looking for a monoidal closed category

$$[\![p]\!] \in \mathcal{L}(A, B) \qquad p : A \multimap B$$

If B is a complete or metrizable space, then so is $\mathcal{L}(A, B)$.

Trickier for A though

$$\mathcal{L}(A \otimes B, C) \simeq \mathcal{L}(A, \mathcal{L}(B, C))$$

- Always true algebraically.

Looking for a monoidal closed category

$$[\![p]\!] \in \mathcal{L}(A, B) \qquad\qquad p : A \multimap B$$

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Trickier for A though

$$\mathcal{L}_{\mathrm{B}}(A \otimes_{\mathrm{B}} B, C) \simeq \mathcal{L}_{\mathrm{B}}(A, \mathcal{L}_{\mathrm{B}}(B, C))$$

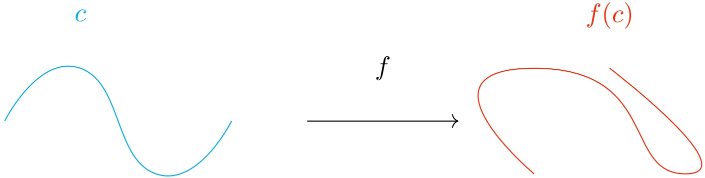
- Always true algebraically.
- Many topological duals E'_β, E'_c, E'_w, E'_μ
- Reflexivity is typically *not* preserved by $\otimes$.
- Not an orthogonality : E'_β is not reflexive.
- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- Monoidal closedness $\Leftrightarrow$ "Grothendieck problème des topologies".
- The solution is often polarized and enforced by important theorems (e.g. Banach-Steinhaus for barrelled spaces [K. 2020]).

Convenient vector spaces

A first interpretation of Higher-Order Smooth Functions.

$$\mathcal{C}^\infty(A \times B, C) \simeq \mathcal{C}^\infty(A, \mathcal{C}^\infty(B, C))$$

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.



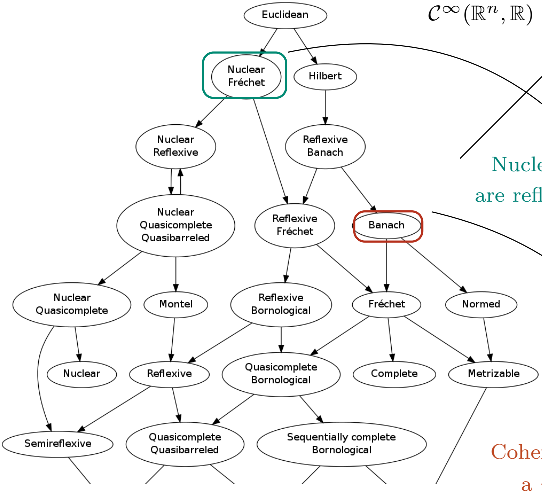
A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve. faire un dessin

The convenient setting of global analysis, Kriegl, Michor (1997)

A convenient differential category Blute, Ehrhard, Tasson(2012)

But we are missing an important notion ...

Looking for reflexivity



$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

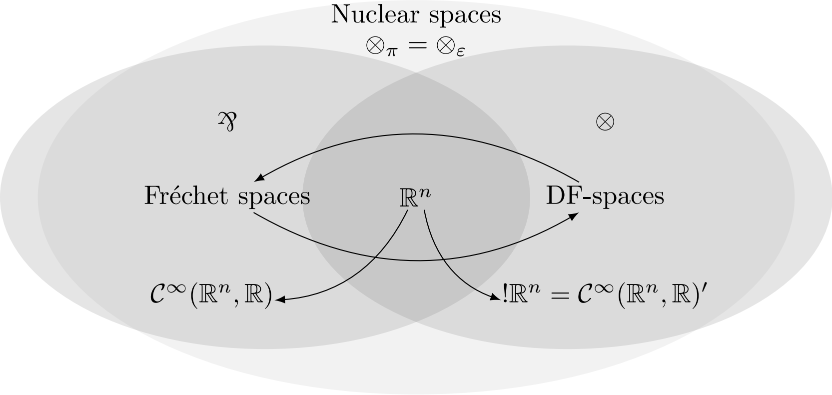
Nuclear Fréchet spaces
are reflexive and complete

Coherent Banach spaces, Girard 2004,
a *norm* is too restrictive

Let us take the other way around, through Nuclear Fréchet spaces.

A proper setting for ! as a distribution space

The analogy between $!A$ and spaces of distributions can be made formal in a model made of metrisable complete spaces (aka Fréchet spaces), and their duals.



Distributions and functions

$$\mathcal{C}^\infty(E, F)'$$

- **Polarization** is back.
  - $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is always Fréchet and $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is DF.
  - If F is Fréchet, then $\mathcal{C}^\infty(\mathbb{R}^n, F)$ is Fréchet
- The importance of **Seely's isomorphism**
  - For linear maps: $\mathcal{L}_\beta(\hat{E}, \mathcal{L}_\beta(F, G)) \simeq \mathcal{L}_\beta(\widehat{E \otimes_\beta F}, G)$
  - From linear maps to smooth maps: **Schwartz' Kernel Theorem**

$$\mathcal{C}^\infty(E, \mathbb{K})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{K})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{K})'$$

But we're still missing higher-order !

A logical account for LPDE K., 2018

Distributions and functions

$$\mathcal{C}^\infty(E, F)'$$

- **Polarization** is back.
  - $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is always Fréchet and $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is DF.
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But we're still missing higher-order !

If E is Nuclear DF, then $\mathcal{C}^\infty(E, F)$ can be defined to be Fréchet thanks to **a quantitative refinement**.

A logical account for LPDE K., 2018

Les fonctions parlent aux fonctions

Gannoun, Hachaichi, Ouerdiane, et Rezgui, Un théorème de dualité entre espaes de fonctions holomorphes à croissance exponentielles, 1999

Higher-order functions for Banach spaces:

$$Exp_{\theta,m}(B) := \{f : B \to \mathbb{C}, \forall m, \exists K, \forall z, |f(z)| \leq K e^{\theta(m||z||)}\}$$

$$!_{\theta,m} B := Exp_{\theta,m}(B)'$$

issue: $||z||$: B needs to be normed.

Higher-order functions for Fréchet spaces: The power of Fréchet spaces comes from their descriptions as a countable projective limit of Banach spaces:

$$N := \varprojlim_{p} N_p \qquad N' := \varinjlim_{p} N'_p$$

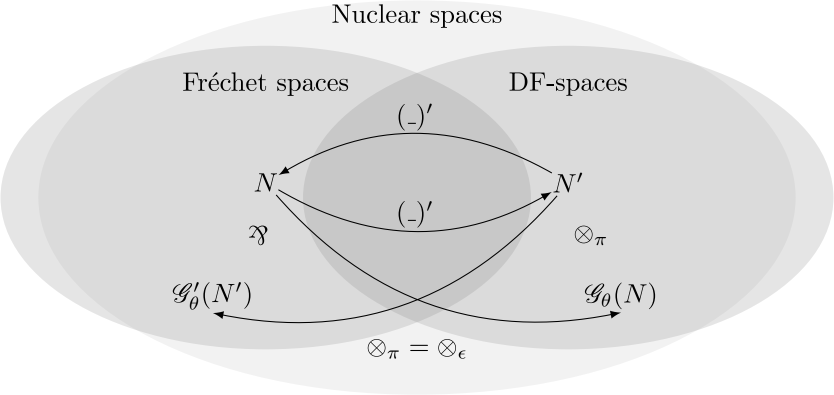
$$\mathscr{F}_\theta(N') = \varprojlim_{m,p} \mathrm{Exp}(N'_p, \theta, m) \qquad \mathscr{G}_\theta(N) = \varinjlim_{m,p} \mathrm{Exp}(N_p, \theta, m).$$

$$!_{\theta^-} F' := \mathscr{F}_\theta(N')' \qquad !_{\theta^+} F := \mathscr{G}_\theta(N)'$$

Les fonctions parlent aux fonctions

The space of Young functions is a space of indices and endowed with a the convex duality operation $(\_)^\star$.

$$\Theta : \{\theta, +, (\_ \times e^{\text{-}}), (\_)^\star\}$$



There is a lot to unpack in this model, in terms of graded logic, of polarization, and in terms of the newly introduced **co-digging**

Making (categorical models of) DiLL symmetric

Higher-Order via promotion

Exponential rules of Linear Logic (Resources)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

Exponential rules added by Differential Linear Logic (Distributions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \quad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}}$$

The promotion rule $\mathsf{p} : !A \to !!A \quad \delta_x \mapsto \delta_{\delta_x}$**:**

- Makes $(!, \mathsf{d}, \mathsf{p})$ a co-monad : $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- Is a co-monoidal operation on $!A$: $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- The cut-elimination between p and $\bar{\mathsf{d}}$ express the chain rule:

$$D_0(g \circ f) = D_{f(0)} g \circ D_0 f \qquad \bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$$

Codigging

Exponential rules of Linear Logic (Resources or functions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

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Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- $\bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$

..

Co-digging ? $\bar{\mathsf{p}} : !!A \to !A$**:**

- $\bar{\mathsf{d}}; \bar{\mathsf{p}} = \mathsf{id}$
- $\bar{\mathsf{c}}; \bar{\mathsf{p}} = \bar{\mathsf{p}} \otimes \bar{\mathsf{p}}; \bar{\mathsf{c}}$
- $\bar{\mathsf{p}}; \mathsf{d} = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$

Codigging

Exponential rules of Linear Logic (Resources or functions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

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..

Co-digging ? $\bar{\mathsf{p}} : !!A \to !A$**:** $g : !A \Rightarrow !A$

- $\bar{\mathsf{d}}; \bar{\mathsf{p}} = \mathsf{id}$ $\quad \mathrm{D}_0(g) = id.$
- $\bar{\mathsf{c}}; \bar{\mathsf{p}} = \bar{\mathsf{p}} \otimes \bar{\mathsf{p}}; \bar{\mathsf{c}}$ $\quad g(x+y) = g(x) * g(y)$
- $\bar{\mathsf{p}}; \mathsf{d} = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$

The missing rule of Differential Linear Logic

Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p};\mathsf{d} = \mathsf{id}$.
- $\mathsf{p};\mathsf{c} = \mathsf{c};\mathsf{p} \otimes \mathsf{p}$
- $\bar{\mathsf{d}};\mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}};\mathsf{p} \otimes \bar{\mathsf{d}};\bar{\mathsf{c}}$

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- $\bar{\mathsf{p}};\mathsf{d} = \mathsf{c};\bar{\mathsf{p}} \otimes \mathsf{d};\mathsf{w} \otimes \mathsf{d}$?

Implicit definition of the exponential:

$$g = exp^* : \phi \mapsto \sum_n \frac{1}{n!}\phi^* \qquad \bar{\mathsf{p}} : \delta_\phi \mapsto \sum_n \frac{1}{n!}\phi^{*^n}$$

The co-chain rule:

$$\bar{\mathsf{p}}(\delta_\phi)(\ell) = \sum_{n \geq 0} \frac{1}{n!}\phi^{*^n}(\ell) = \cdots = \mathsf{c};\bar{\mathsf{p}} \otimes \mathsf{d};\mathsf{w} \otimes \mathsf{d}$$

The monadic rules:

$$!\bar{\mathsf{d}};\bar{\mathsf{p}} = id \qquad \forall v, \bar{\mathsf{p}}(\delta_{D_0(\_)(v)}) = \delta_v \qquad \forall v, \forall f, \sum_n \frac{1}{n!}\mathrm{D}_0^{(n)} f(v) = f(v)$$

Landing back on **Taylor expansion within the logic** !

Taylor Expansion as a Monad in Models of DiLL K., Lemay, 2023

The quantitative monad

A **monad** over a type A:

- It *encapsulate* a certain kind of values: $u_A : A \to M(A)$.
- It allows *computation* on these values: $\mu_A : M(M(A)) \to M(A)$.

Examples:

- Partiality: $M : A \mapsto A + \perp$, $u_A : a \mapsto a$
- Non-determinism: $M : A \mapsto \mathcal{P}(A)$, $u_A : a \mapsto \{a\}$
- Effect: $M : A \mapsto (S \to (A \times S))$, $u_A : a \mapsto (s \mapsto (a, s))$

Co-digging gives us a **monadic characterization of Taylor expansion**. The following form a monad in quantitative models of λ-calculus.

- $M : E \to \mathcal{C}^\infty(E, \mathbb{K})'$
- $u : v \mapsto (f \mapsto D_0(f)(v))$
- $\mu : \delta_\phi \mapsto \sum \frac{1}{n!} \phi^{*^n}$

Laplace transformation and grading in DiLL

A reason for this symmetry

(Categorical models of) DiLL is a very symmetric structure. Why ?

Exponential connectives:

$$[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{K})' \qquad [\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{K})$$

The Laplace transform: is widely use in calculus as in turns *differential equations* into *polynomial equations*.

$$\mathscr{L} : (f : \mathbb{R} \to \mathbb{R}) \mapsto (x : \mathbb{R}) \mapsto \int_0^\infty f(t) e^{-xt} dt$$

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$$\mathscr{L} : (f : \mathbb{R} \to \mathbb{R}) \mapsto (x : \mathbb{R}) \mapsto \int_0^\infty f(t) e^{-xt} dt$$

That's not very higher-order

$$\mathscr{L} : \begin{cases} !E & \to ?E \\ \phi & \mapsto ((\ell : E') \mapsto \phi((y : E) \mapsto (e^{<\ell|y>} : \mathbb{K}))) \end{cases}$$

Let's get back to basic mathematics

$$\mathscr{L} : \begin{cases} !E & \to ?E \\ \phi & \mapsto ((\ell : E') \mapsto \phi((y : E) \mapsto e^{<\ell|y>})) \end{cases}$$

Let's consider ℓ an element of $A' = \mathcal{L}(A, \mathbb{R})$, acting on elements t of A.

From co-weakening to weakening

$$\mathscr{L}(\overline{\mathsf{w}}_A)(r) = \ell \mapsto \delta_0(t \mapsto e^{\ell(t)}) = \ell \mapsto r \cdot e^0 = \ell \mapsto r = \mathsf{w}(r)$$

From co-dereliction to dereliction

$$\mathscr{L}(\overline{\mathsf{d}}_A)(y) = \ell \mapsto D_0(t \mapsto e^{\ell})(y) = \ell \mapsto \ell(y) = \mathsf{d}(y)$$

From co-contraction to contraction

$$\begin{aligned} \mathscr{L}(\overline{\mathsf{c}}_A(\phi \otimes \psi)) &= \ell \mapsto (\phi * \psi)(t \mapsto e^{\ell(t)}) = \ell \mapsto \phi\left(s \mapsto \psi(t \mapsto e^{\ell(t+s)})\right) \\ &= \ldots \text{ linearity of } \ell, \phi, \text{ and } \psi \\ &= \mathsf{c}(\mathscr{L}(\phi) \otimes \mathscr{L}(\psi)) \end{aligned}$$

Laplace distributors in differential categories

Theorem

In categorical models of DiLL and in concrete models of it:

$$\mathscr{L}(\overline{\mathsf{w}}, \overline{\mathsf{c}}, \overline{\mathsf{d}}, \overline{\mathsf{p}}) = \mathsf{w}, \mathsf{c}, \mathsf{d}, \mathsf{p}$$

In categorical models of DiLL, one can define a 1**-exponential map** is a map $\mathsf{e} : !I \to I$ such that:

- $\overline{\mathsf{c}}_1 ; e = e \otimes e$,
- $\overline{\mathsf{w}}_1 ; e = \mathrm{id}$,
- $\overline{\mathsf{d}}_1 ; e = \mathrm{id}$.

Define the map $\mathscr{L}_A : !A \to ?A^\perp$ as the curry of the following composite:

$$(!A)^\perp \otimes !A \xrightarrow{\Phi} !(A^\perp \otimes A) \xrightarrow{\epsilon} !1 \to 1.$$

Then $\mathscr{L}$ is a Laplace distributor.

Laplace Distributors and Laplace Transformations for Differential Categories, K., Lemay, 2024

Mixing grading and differentiation : back to the syntax

From co-digging to co-promotion ?

This picture sadly doesn't work, as the cut-elimination rule between p and $\bar{\mathsf{p}}$ is nowhere to be found.

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p$$

Linear Logic

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c}$$

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}} \qquad \boxed{\frac{?\Gamma \vdash x : A}{?\Gamma \vdash e^{x|\_} : ?A}\, \bar{p}}$$

Differential Linear Logic

This needs to be made **polarized** and **graded** for cut-elimination to work : closer to the model.

Mixing DiLL with grading: Laplace needs to intervene

If Laplace acts on structural rules, it needs to acts on indices of structural rules too.

$$\begin{tikzcd}
?_{x^-} A & & !_{x^+} A \arrow[ll, "\mathscr{L}"'] \arrow[d, "(_)^{\perp}", "\overline{\mathsf{w}}, \overline{\mathsf{c}}, \overline{\mathsf{d}}, \overline{\mathsf{p}}"'] \\
!_{x^-} A \arrow[u, "(_)^{\perp}"', "\mathsf{w}, \mathsf{c}, \mathsf{d}, \mathsf{p}"] \arrow[rr, "\mathscr{L}"'] & & ?_{x^+} A
\end{tikzcd}$$

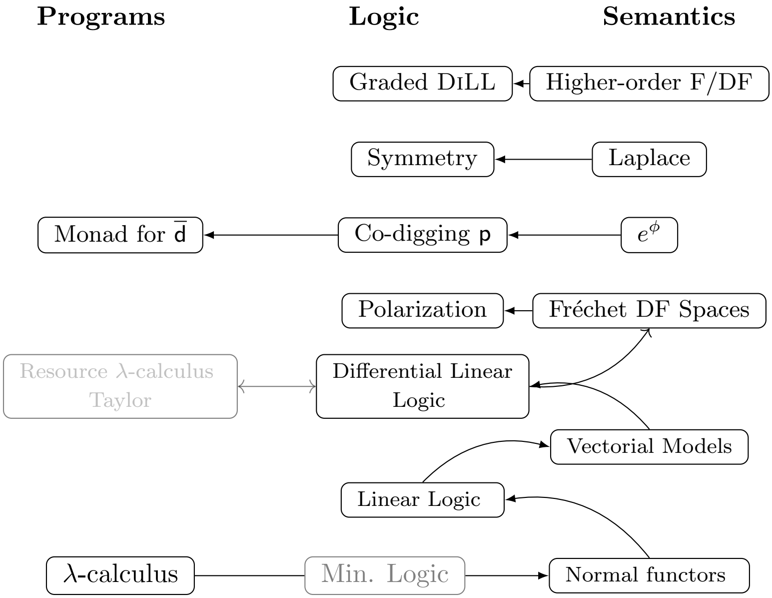
$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_{0^-} A}\,\mathsf{w} \qquad \frac{}{\vdash !_{0^+} A}\,\overline{\mathsf{w}} \qquad \frac{\vdash \Gamma, ?_{x^-} A, ?_{y^-} A}{\vdash \Gamma, ?_{x^- + y^-} A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, !_{x^+} A \qquad \vdash \Delta, !_{y^+} A}{\vdash \Gamma, \Delta, !_{x^+ + y^+} A}\,\overline{\mathsf{c}}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?_{1^-} A}\,\mathsf{d} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_{1^+} A}\,\overline{\mathsf{d}} \qquad \frac{\vdash \Gamma, ?_{x^-} A \qquad x \leq y}{\vdash \Gamma, ?_{y^-} A}\,\mathsf{d}_I$$

$$\frac{\vdash \Gamma, !_{x^+} A \qquad x \leq y}{\vdash \Gamma, !_{y^+} A}\,\overline{\mathsf{d}}_I \qquad \frac{\vdash ?_{X^-} \mathcal{A}, B \quad y \neq 0}{\vdash ?_{y^- \times X^-} \mathcal{A}, !_{y^-} B}\,\mathsf{p} \qquad \frac{\vdash !_{x^+} A, B \qquad y \neq 0}{\vdash !_{y^+ \times x^+} A, ?_{y^+} B}\,\overline{\mathsf{p}}$$

Double indexed differential linear logic: reconciling resources and differential operators, K., Mirwasser, 2025

Our Journey



Programs
Logic
Semantics
Graded DiLL
Higher-order F/DF
Symmetry
Laplace
Monad for $\bar{\mathsf{d}}$
Co-digging p
e^{ϕ}
Polarization
Fréchet DF Spaces
Resource λ-calculus Taylor
Differential Linear Logic
Vectorial Models
Linear Logic
λ-calculus
Min. Logic
Normal functors

Take-home messages

- Linear Logic restricts contraction and weakening to specific formulas: this refines conjunction and disjunction to connectives with different reversibility properties.

- Linear Logic and its fragments bring insight about resource usage and differentiation to computer science.

- A successful methodology for and from Linear Logic consists in using the semantics to enrich the syntax. One can constructs discrete models close to the computations but also attach Linear Logic to more functional analysis concepts.

Thank you for listening !