The rise of quantitative category theory

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Dept. of Computer Science

Topos Colloquium
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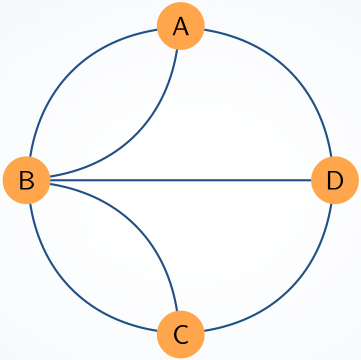
Motivation: Number of steps



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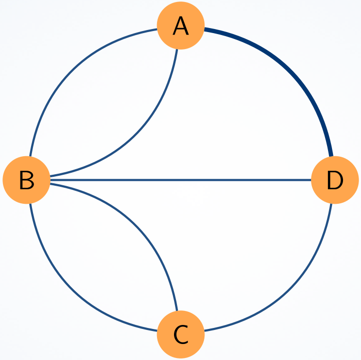

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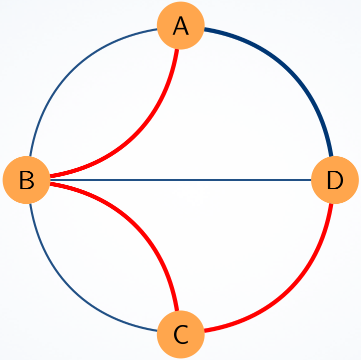

Motivation: Limits and approximations

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots \longrightarrow \operatorname{colim}_i X_i$$

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$$\begin{tikzcd}
X_0 \arrow[r] \arrow[drrrrr, bend right] & X_1 \arrow[r] \arrow[drrrr, bend right] & X_2 \arrow[r] \arrow[drrr, bend right] & X_3 \arrow[r] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
& & & & & Y
\end{tikzcd}$$

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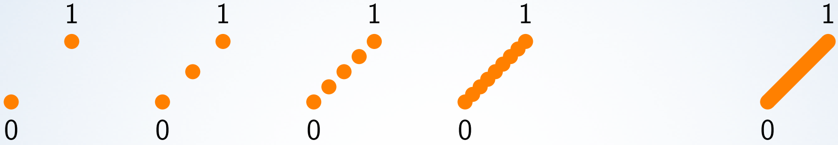

$$
\begin{tikzcd}
X_0 \arrow[r] \arrow[rrrrrd, bend right] & X_1 \arrow[r] \arrow[rrrrd, bend right] & X_2 \arrow[r] \arrow[rrrd, bend right] & X_3 \arrow[r] \arrow[rrd, bend right] & \cdots \arrow[r] \arrow[rd, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
 & & & & & Y
\end{tikzcd}
$$

Motivation: Lenses

$$
\begin{tikzcd}
\text{Montreal} \arrow[r] \arrow[d, dashed, shift right] & \text{New York} \arrow[dr] \arrow[d, dashed, shift right] & \text{Edinburgh} \arrow[d, dashed, shift right] \\
\text{Toronto} \arrow[ur] \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] & \text{San Francisco} \arrow[r] \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] & \text{London} \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] \\
\text{Vancouver} \arrow[r] \arrow[u, dashed, shift right] & \text{Seattle} \arrow[ur] \arrow[u, dashed, shift right] & \text{Liverpool} \arrow[u, dashed, shift right]
\end{tikzcd}
$$

$$
\text{Canada} \longrightarrow \text{United States} \longrightarrow \text{United Kingdom}
$$

Weighted categories and functors

Main definitions

A *weighted category* is a category where each morphism $f : X \to Y$ is equipped with a nonnegative number $w(f)$ called the *weight*, such that

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$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z \end{tikzcd} \qquad w(g \circ f) \leq w(f) + w(g).$$

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A *weighted functor* is a functor $F : \mathsf{C} \to \mathsf{D}$ such that for every morphism f of C,

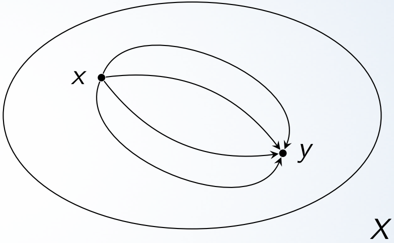
$$w(Ff) \leq w(f).$$

Examples

Categories of paths *of a space*

Let X be a metric space (e.g. $\mathbb{R}^n$). The weighted category $\mathrm{Path}(X)$ has

- As objects, the points of X;
- As morphisms, the curves in X with their length as weight.

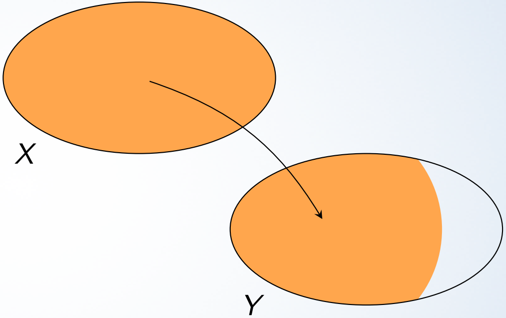


Examples

Categories *of spaces*

Let X and Y be metric spaces, and let $f : X \to Y$. The *density* of f is

$$\sup_{y \in Y} d\big(f(X), y\big).$$



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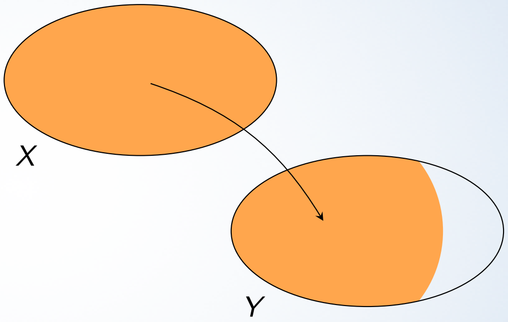
The weighted category MetDens has

- As objects, metric spaces;
- As morphisms, distance-nonincreasing maps with their density.



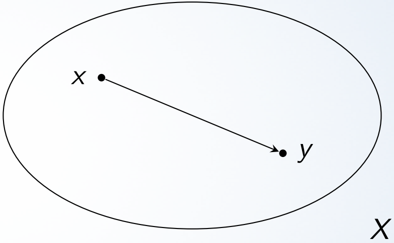
Examples

Generalized metric spaces

A *pseudo-quasi* (or *Lawvere*) *metric space* is a set X with a "cost" function $c : X \times X \to [0, \infty]$ such that

- $d(x, x) = 0$;
- $d(x, z) \le d(x, y) + d(y, z)$

A pq-metric space is a weighted preorder.



Examples

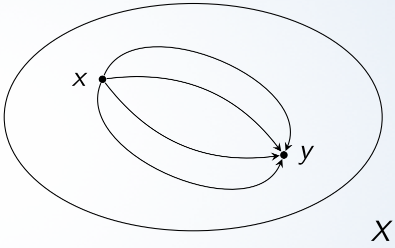
Optimization over paths

Given a weighted category C, for objects X and Y consider the "optimum" weight

$$\inf_{f:X\to Y} w(f)$$

This gives a pq-metric on the objects of C. We call the resulting space Opt(C).

This is secretly widely used!



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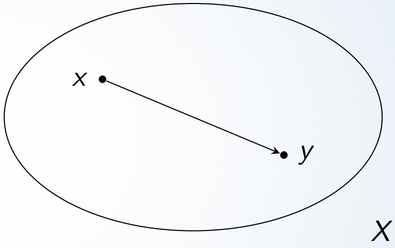
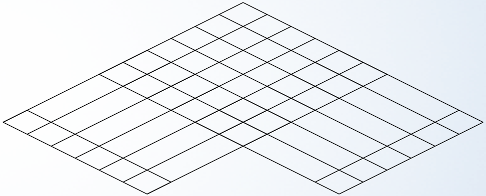


Couplings and optimal transport

Given a (metric) space X, the product space $X \times X$ encodes "transport plans".



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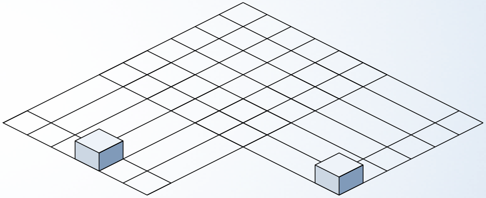

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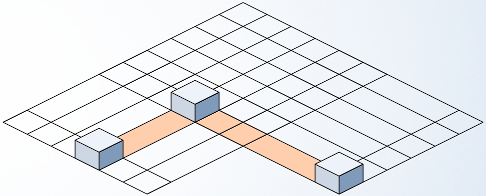

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The category PX has

- As objects, prob. measures p on X;
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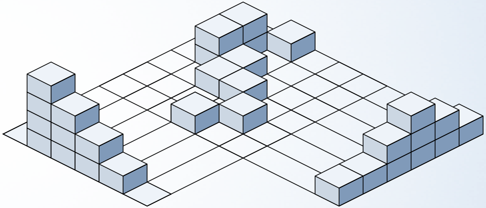

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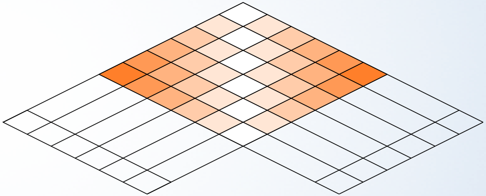

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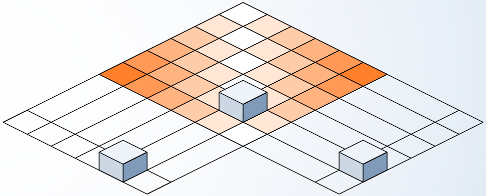

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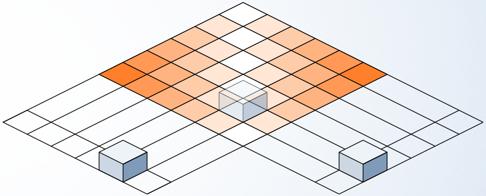
If X has a cost function c,

$$c(x, x) = 0$$



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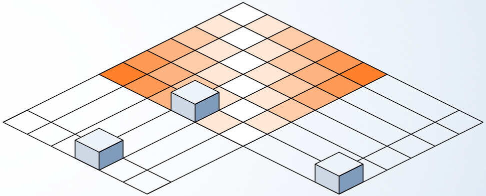

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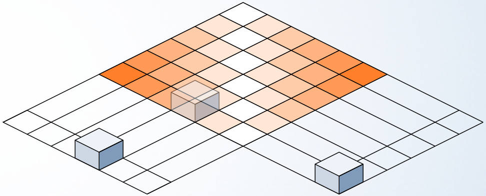
$$c(x, y) = \text{ “cost of transport”}$$



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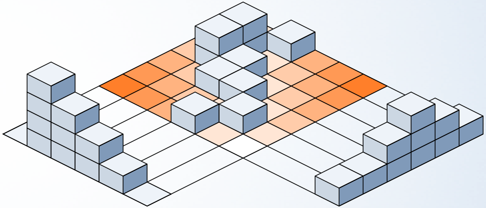
If X has a cost function c,

$$\mathrm{Cost}(t) := \int_{X^2} c(x,y)\, t(dx\, dy).$$



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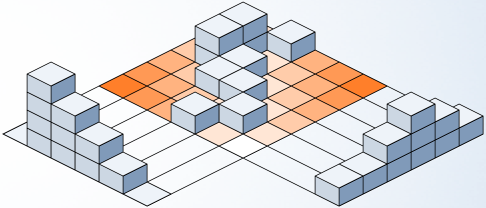
$$\mathrm{Cost}_k(t) := \sqrt[k]{\int_{X^2} c(x,y)^k \, t(dx\, dy)}.$$



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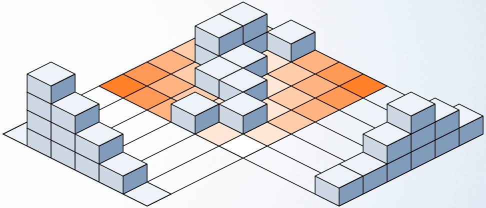
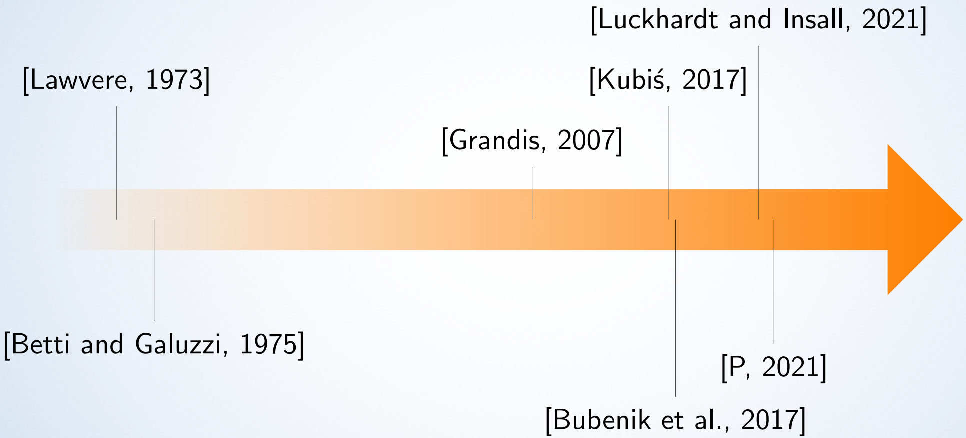
Opt(C) gives the famous *Wasserstein spaces* [Villani, 2009].



A little history



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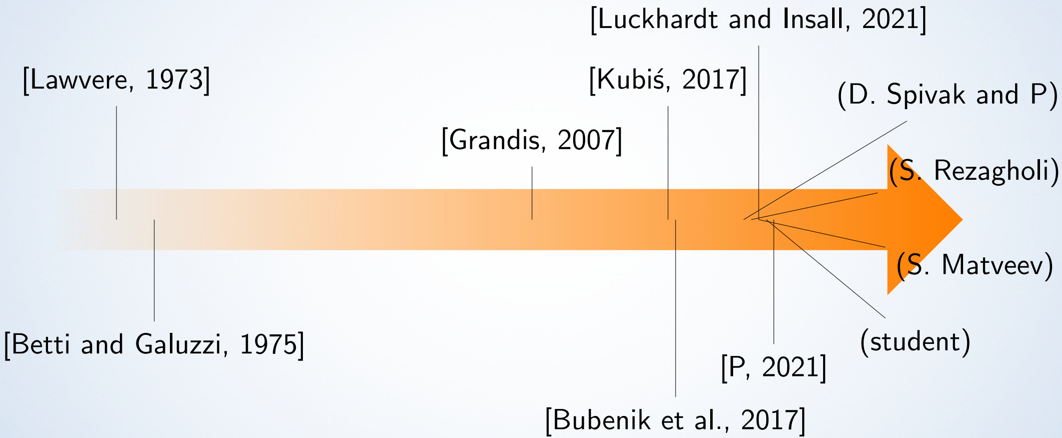
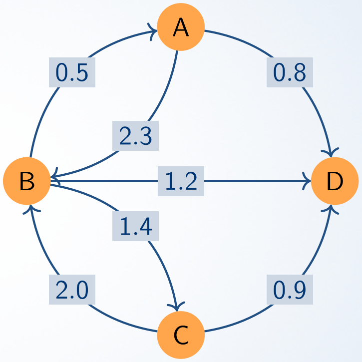


Weighted categories and weighted graphs

Definition: free weighted category

Let G be a weighted graph. The *free weighted category over* G has

- As objects, the vertices of G;
- As morphisms, *walks* in G;
- The weight is the sum of the weights.



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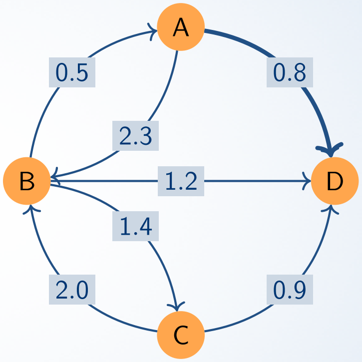

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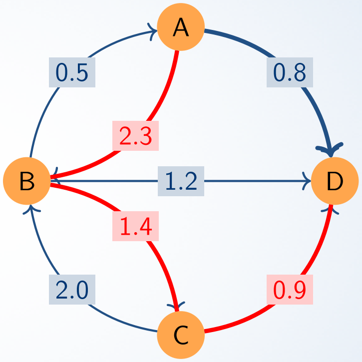

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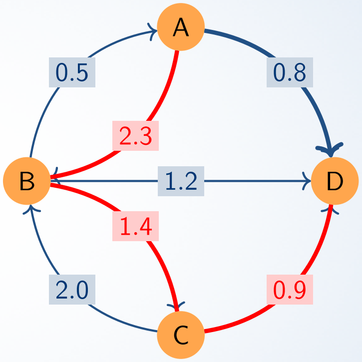

This way, weighted categories are monadic over weighted graphs [Kubiś, 2017].

Weighted* limits (ongoing, with S. Rezagholi)

Definition: weighted* diagrams

Let $D : \mathsf{J} \to \mathsf{C}$ be a diagram.

$$J \xrightarrow{\quad j \quad} J' \quad \rightsquigarrow \quad DJ \xrightarrow[\color{orange}{3}]{\quad Dj \quad} DJ'$$

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- A *co-weighting* is defined analogously, except that

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Let $D : J \to C$ be a diagram with weighting V. A *limit of D weighted by V* is an object L together with a natural, weight-preserving bijection

$$\mathsf{C}(X, L) \cong \mathrm{Cones}_V(X, D),$$

where the V-weight of a cone $\alpha : X \Rightarrow D$ is given by

$$\sup_{J \in \mathsf{J}} \max\big(w(\alpha_J) - VJ, 0\big).$$

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 & X \arrow[d, dashed] \arrow[ddl] \arrow[ddr] & \\
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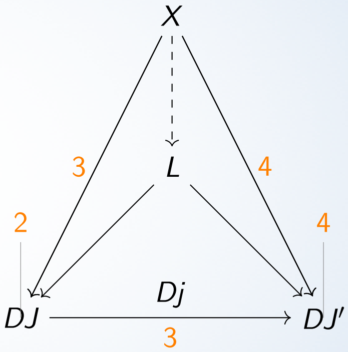

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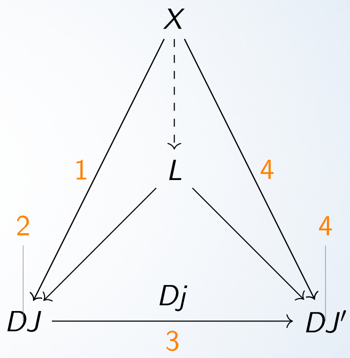

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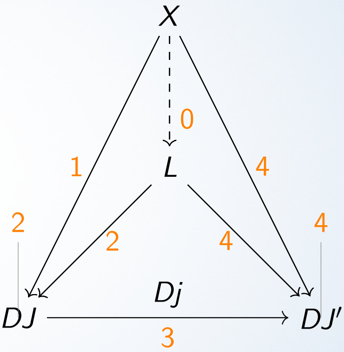

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Let $D : J \to C$ be a diagram with coweighting V. A *colimit of D weighted by V* is an object L together with a natural, weight-preserving bijection

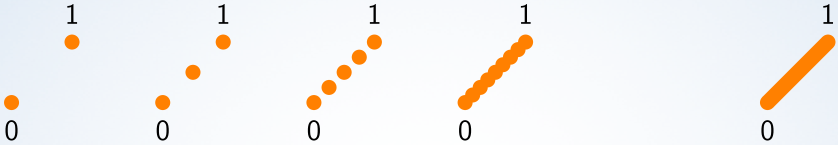
$$\mathsf{C}(L, Y) \cong \mathrm{Cocones}_V(D, Y),$$

where the V-weight of a cocone $\alpha : D \Rightarrow Y$ is given by

$$\sup_{J \in \mathsf{J}} \max\big(w(\alpha_J) - VJ, 0\big).$$

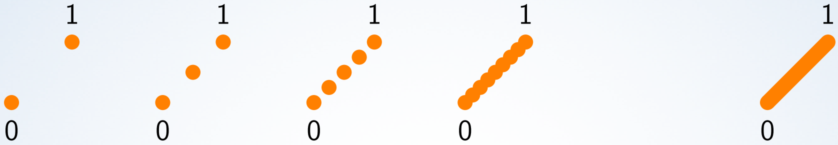
$$\begin{tikzcd}
DJ \arrow[rr, "Dj", "3"'] \arrow[dr, "4"] \arrow[ddr, "4"'] & & DJ' \arrow[dl, "2"'] \arrow[ddl, "1"] \\
& L \arrow[d, dashed, "0"] & \\
& Y &
\end{tikzcd}$$

Example in CMetDens



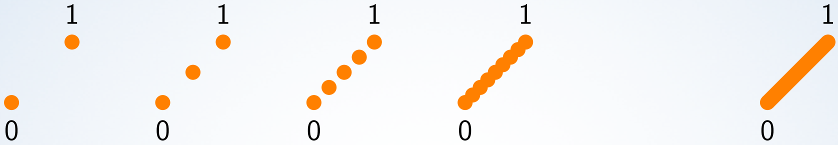
$$
\begin{tikzcd}
X_0 \arrow[r] \arrow[drrrrr, bend right] & X_1 \arrow[r] \arrow[drrrr, bend right] & X_2 \arrow[r] \arrow[drrr, bend right] & X_3 \arrow[r] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
 & & & & & Y
\end{tikzcd}
$$

Example in CMetDens



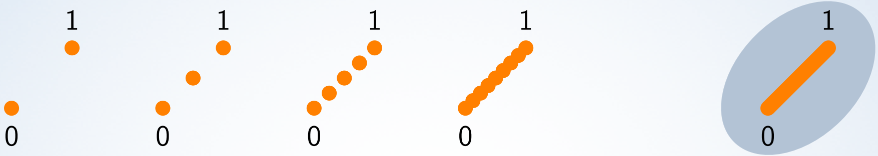
$$
\begin{tikzcd}
X_0 \arrow[r, "1/2"] \arrow[drrrrr, bend right] & X_1 \arrow[r, "1/4"] \arrow[drrrr, bend right] & X_2 \arrow[r, "1/8"] \arrow[drrr, bend right] & X_3 \arrow[r, "1/16"] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
 & & & & & Y
\end{tikzcd}
$$

Example in CMetDens



$$
\begin{tikzcd}
\frac{1}{2} \arrow[d, dash] & \frac{1}{4} \arrow[d, dash] & \frac{1}{8} \arrow[d, dash] & \frac{1}{16} \arrow[d, dash] & & \\
X_0 \arrow[r, "1/2"] \arrow[drrrrr, bend right] & X_1 \arrow[r, "1/4"] \arrow[drrrr, bend right] & X_2 \arrow[r, "1/8"] \arrow[drrr, bend right] & X_3 \arrow[r, "1/16"] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
& & & & & Y
\end{tikzcd}
$$

Example in CMetDens



$$\begin{tikzcd}
1/2 \arrow[d, dash] & 1/4 \arrow[d, dash] & 1/8 \arrow[d, dash] & 1/16 \arrow[d, dash] & & \\
X_0 \arrow[r, "1/2"] \arrow[drrrrr, bend right] & X_1 \arrow[r, "1/4"] \arrow[drrrr, bend right] & X_2 \arrow[r, "1/8"] \arrow[drrr, bend right] & X_3 \arrow[r, "1/16"] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
 & & & & & Y
\end{tikzcd}$$

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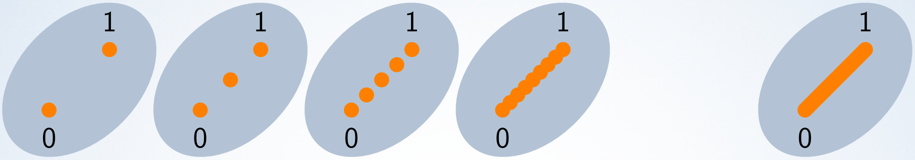
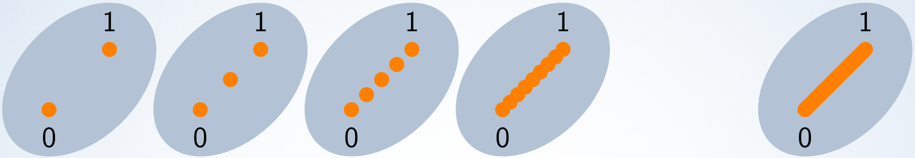


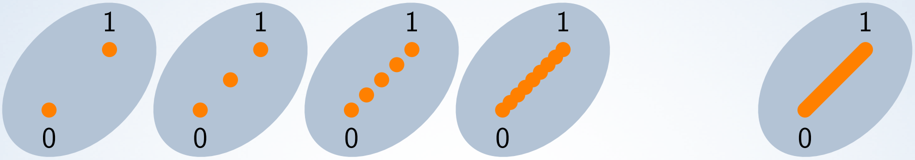
$$
\begin{tikzcd}
\overset{1/2}{X_0} \arrow[r, "1/2"] \arrow[drrrrr, bend right] & \overset{1/4}{X_1} \arrow[r, "1/4"] \arrow[drrrr, bend right] & \overset{1/8}{X_2} \arrow[r, "1/8"] \arrow[drrr, bend right] & \overset{1/16}{X_3} \arrow[r, "1/16"] \arrow[drr, bend right] & \cdots \arrow[r] \arrow[dr, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed] \\
 & & & & & Y
\end{tikzcd}
$$

Example in CMetDens



$$
\begin{tikzcd}
{\color{orange}1/2} \arrow[d, dash] & {\color{orange}1/4} \arrow[d, dash] & {\color{orange}1/8} \arrow[d, dash] & {\color{orange}1/16} \arrow[d, dash] & & \\
X_0 \arrow[r, "{\color{orange}1/2}"] \arrow[rrrrrd, bend right, "{\color{orange}1/2}"'] & X_1 \arrow[r, "{\color{orange}1/4}"] \arrow[rrrrd, bend right, "{\color{orange}1/4}"'] & X_2 \arrow[r, "{\color{orange}1/8}"] \arrow[rrrd, bend right, "{\color{orange}1/8}"'] & X_3 \arrow[r, "{\color{orange}1/16}"] \arrow[rrd, bend right, "{\color{orange}1/16}"'] & \cdots \arrow[r] \arrow[rd, bend right] & \operatorname{colim}_i X_i \arrow[d, dashed, "{\color{orange}0}"] \\
& & & & & Y
\end{tikzcd}
$$

Example in CMetDens



$$
\begin{tikzcd}
1/2 & 1/4 & 1/8 & 1/16 & & \\
X_0 \arrow[r, "1/2"] \arrow[rrrrrd, bend right, "\geq 1/2"'] & X_1 \arrow[r, "1/4"] \arrow[rrrrd, bend right, "\geq 1/4"'] & X_2 \arrow[r, "1/8"] \arrow[rrrd, bend right, "\geq 1/8"'] & X_3 \arrow[r, "1/16"] \arrow[rrd, bend right, "\geq 1/16"'] & \cdots \arrow[r] \arrow[rd, bend right] & \mathrm{colim}_i X_i \arrow[d, dashed, "> 0"] \\
 & & & & & Y
\end{tikzcd}
$$

Weighted lenses (P 2021)

$$\begin{tikzcd}
\text{Montreal} \arrow[r] \arrow[d, dashed, shift right] & \text{New York} \arrow[d, dashed, shift right] \arrow[dr] & \text{Edinburgh} \arrow[d, dashed, shift right] \\
\text{Toronto} \arrow[u, dashed, shift right] \arrow[ur] \arrow[d, dashed, shift right] & \text{San Francisco} \arrow[u, dashed, shift right] \arrow[r] \arrow[d, dashed, shift right] & \text{London} \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] \\
\text{Vancouver} \arrow[u, dashed, shift right] \arrow[r] & \text{Seattle} \arrow[u, dashed, shift right] \arrow[ur] & \text{Liverpool} \arrow[u, dashed, shift right] \\
\text{Canada} \arrow[r] & \text{United States} \arrow[r] & \text{United Kingdom}
\end{tikzcd}$$

Weighted lenses (P 2021)

$$
\begin{tikzcd}
\text{Montreal} \arrow[rr, "1"] \arrow[d, dashed, shift right] & & \text{New York} \arrow[rr, "2"] \arrow[d, dashed, shift right] & & \text{Edinburgh} \arrow[d, dashed, shift right] \\
\text{Toronto} \arrow[urr, "1"] \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] & & \text{San Francisco} \arrow[rr, "2"] \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] & & \text{London} \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] \\
\text{Vancouver} \arrow[rr, "1"] \arrow[u, dashed, shift right] & & \text{Seattle} \arrow[urr, "2"] \arrow[u, dashed, shift right] & & \text{Liverpool} \arrow[u, dashed, shift right]
\end{tikzcd}
$$

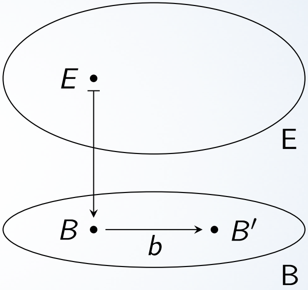
$$
\begin{tikzcd}
\text{Canada} \arrow[r, "1"] & \text{United States} \arrow[r, "2"] & \text{United Kingdom}
\end{tikzcd}
$$

Weighted lenses (P 2021)

Definition: weighted lifting

- Let $F : \mathsf{E} \to \mathsf{B}$ be a functor;
- Let $b : B \to B'$ be a morphism of B;
- Let $E \in \mathsf{E}$ with $F(E) = B$.

A *lifting of b at E* consists of



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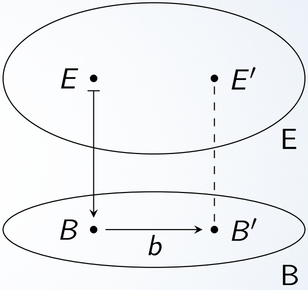
- an object $E' \in \mathsf{E}$ with $F(E') = B'$;



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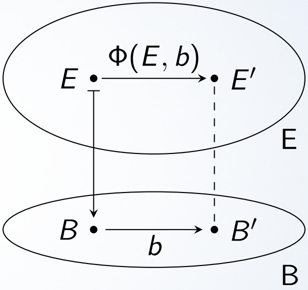
A *lifting of b at E* consists of

- an object $E' \in \mathsf{E}$ with $F(E') = B'$;
- an arrow $\Phi(E, b) : E \to E'$ of E such that $F\big(\Phi(E, b)\big) = b$, with the same weight as b.

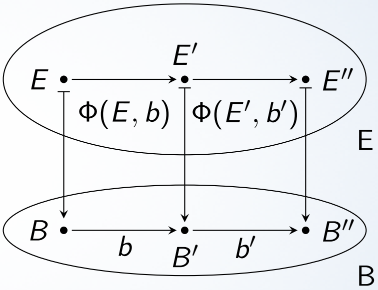


Weighted lenses (P 2021)

Definition: weighted lens

Let E and B be categories. A *weighted lens* from E $\to$ B consists of

- A functor F : E $\to$ B;
- For each morphism $b : B \to B'$ of B and each object E of E with $F(E) = B$, a chosen weighted lifting $\Phi(E, b) : E \to E'$, such that
- The identities are lifted to identities;
- The choice of liftings preserves composition.



Lenses between categories of couplings

Theorem (P 2021)

Let X and Y be pq-metric, standard Borel spaces. Let (f, ϕ) be a weighted lens such that the assignments f and ϕ, on objects, are measurable (for example, a product projection).

There is a weighted lens $(f_\sharp, \tilde{\varphi}_\sharp)$ between PX and PY where

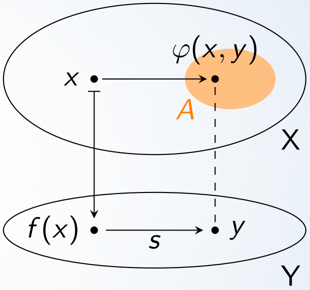
- The projection $f_\sharp : PX \to PY$ is the usual pushforward of measures;
- The lifting $\tilde{\varphi}_\sharp : PX \times_{PY} P(Y \times Y) \to P(X \times X)$ takes $p \in PX$ and a coupling $s \in P(Y \times Y)$ whose first marginal has to equal $f_* p$, and returns the coupling $\tilde{\varphi}_\sharp(p, s) \in P(X \times X)$ given, for all measurable subsets $A, A' \subseteq X$, by

$$\tilde{\varphi}_\sharp(p, s)(A \times A') := \int_A \int_Y 1_{A'}\big(\varphi(x, y)\big)\, s\big(dy|f(x)\big)\, p(dx).$$

Lenses between categories of couplings

This is a "lifting of random transitions".

$$\tilde{\varphi}_\sharp(s)(A|x) = \int_Y 1_A\big(\varphi(x,y)\big)\, s\big(dy|f(x)\big)$$



Conclusion

- Weighted categories combine category theory with quantitative reasoning.
- They have been around for almost 50 years, but only now researchers are starting to recognize their potential.
- They seamlessly integrate with weighted graphs, metric geometry, and probability.
- They are secretly used already in several mathematical fields.

Conclusion

- Weighted categories combine category theory with quantitative reasoning.
- They have been around for almost 50 years, but only now researchers are starting to recognize their potential.
- They seamlessly integrate with weighted graphs, metric geometry, and probability.
- They are secretly used already in several mathematical fields.

A request for the community:

I am collecting material on weighted categories, past and present, in order to organize what we have so far. If you have done work with them, please let me know!

Some references

Betti, R. and Galuzzi, M. (1975).
Categorie normate.
Bollettino dell'Unione Matematica Italiana, 11(1):66–75.

Bubenik, P., de Silva, V., and Scott, J. (2017).
Interleaving and Gromov-Hausdorff distance.
arXiv:1707.06288.

Clarke, B. (2020a).
Internal lenses as functors and cofunctors.
In *Proceedings of Applied Category Theory 2019*, volume 323 of *Electronic Proceedings in Theoretical Computer Science*, pages 183–185.

Clarke, B. (2020b).
Internal split opfibrations and cofunctors.
Theory and Applications of Categories, 35(44):1608–1633.

Grandis, M. (2007).
Categories, norms and weights.
Journal of homotopy and related structures, 2(2):171–186.

Kubiś, W. (2017).
Categories with norms.
arXiv:1705.10189.

Lawvere, W. (1973).
Metric spaces, generalized logic and closed categories.
Rendiconti del seminario matematico e fisico di Milano, 43.

Luckhardt, D. and Insall, M. (2021).
Norms on categories and analogs of the Schröder-Bernstein theorem.
arXiv:2105.06832.

Perrone, P. (2021).
Lifting couplings in Wasserstein spaces.
arXiv:2110.06591.

Villani, C. (2009).
Optimal transport: old and new, volume 338 of *Grundlehren der mathematischen Wissenschaften*.
Springer.

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- Lenses

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- Examples
- Couplings and optimal transport

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Weighted categories and weighted graphs

Weighted* limits (ongoing, with S. Rezagholi)

Weighted lenses (P 2021)

- Lenses between categories of couplings

Conclusion

References